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On Covering Prosperities via Neutrosophic e -Open Set in Neutrosophic Topological Space

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ABSTRACT

This research presents and explores a novel category of compactness in neutrosophic topological spaces (NTSs), termed neutrosophic e -compact and neutrosophic locally e -compact. This category is positioned within the frameworks of neutrosophic δ -semicompactness and neutrosophic δ -precompactness, while also encompassing neutrosophic β -compactness. We have derived multiple preservation properties and characterizations associated with neutrosophic e -compactness, along with an examination of its images and preimages across various functions. Following an introduction to the fundamental concepts of neutrosophic sets and NTSs, we define e -open sets, e -continuity, and neutrosophic e_* -continuous in the context of neutrosophic theory, alongside other related findings regarding e -continuity. Additionally, we have identified several preservation properties and characterizations pertinent to e -compactness such as a neutrosophic e -base and a neutrosophic e -subbase.

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1 | Historical Background

Fuzzy set theory, introduced by Zadeh in 1965, has been widely used to deal with uncertainty and vagueness in various fields such as operation research, information theory [1], and control theory [2, 3]. Later on, Atanassov [4] defined intuitionistic fuzzy topological spaces, which is a generalization of fuzzy topological spaces and intuitionistic fuzzy topological spaces. Coker [5, 6] presented the notion of an intuitionistic fuzzy topology based on Chang's sense.

Chandrasekar et al. [7] in 1997 expand this idea to intuitionistic fuzzy spaces and also the notions of intuitionistic fuzzy e -open sets, intuitionistic fuzzy e -continuity, and intuitionistic fuzzy e -compactness and investigate their properties thoroughly. An intuitionistic fuzzy e -open set is the stronger of the two and weaker than an intuitionistic fuzzy δ -pre open set, intuitionistic

fuzzy δ -semi open set (N δ SOs), and stronger than an intuitionistic fuzzy β -open set.

The neutrosophic system was introduced by Smarandache in the early 20th century, and in recent publications, it has established a foundational framework for a broad spectrum of new mathematical theories that expand upon both fuzzy and classical systems. These theories find extensive real-world applications in various domains, including decision-making, medicine, applied mathematics, information systems, and computer science, as well as quadripartitioned and pentapartitioned neutrosophic topology (NT). Smarandache adapted the concepts of intuitionistic fuzzy sets and different types of sets to create neutrosophic sets (NSs), commonly referred to as NSs.

The concept of NT was initially introduced by Smarandache and Shumrani, focusing on nongeneral and standard intervals [8, 9].

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Since that time, several authors have built upon this idea for different applications (see [3, 10–14]). Recently, Al-Omeri and Smarandache [15] investigated continuity in NT. Furthermore, Al-Omeri and other researchers [16] have proposed various applications within neutrosophic theory, especially emphasizing neutrosophic graphs, which include a relevant application regarding earthquake response centers in Japan, as mentioned in [17–19].

Covering properties in neutrosophic topological spaces (NTSs) using neutrosophic e -open sets studied and defined by [20]. Also, they introduce and define neutrosophic e -compactness and neutrosophic e -open cover.

The notion of δ -open sets was introduced by Smets [1] in the field of topological spaces. Subsequently, Vadivel and John Sunda [11] defined δ -open sets specifically for NTSs in 2021. Additionally, Ekici [21] discussed the idea of e -open sets within general topology. This principle has been investigated and broadened by various authors in different topological settings [18]. In 2014, Seenivasan and Kamala [22] introduced the concept of fuzzy e -open sets in a topological space, in conjunction with fuzzy e -continuity. Later, Vadivel et al. [23] examined neutrosophic e -open sets in the framework of NTSs.

The majority of previous research in NT, including studies by Salama et al. [24], emphasizes the generalization of classical topological concepts (for instance, neutrosophic continuity, neutrosophic closed sets, and neutrosophic compactness). However, these studies do not consider external compactness (or e -compactness), which represents a broader form of compactness that eases the criteria set by open cover-based definitions and can illustrate varying degrees of “compactness behavior” in intricate uncertain systems.

In the work by Vadivel et al. [25], the concepts of neutrosophic e -open sets and neutrosophic e -continuity were presented. A new idea based on e -compactness in fuzzy and generalized topological spaces, which proves to be more powerful than the concept of neutrosophic compactness [26]. This study explores the preservation of neutrosophic e -compactness under different mappings (such as continuous and e -continuous functions [27]), a topic that has not been addressed in the existing neutrosophic literature. Additionally, it offers characterizations and examples that illustrate the distinct behavior of neutrosophic e -compactness in comparison to established compactness concepts, defining its fundamental characteristics and connections with various forms of compactness. We conclude by introducing the concept of neutrosophic locally e -compactness and presenting several findings related to it.

The following aspects are addressed in this discussion:

1. In Section 2, we introduce the basic definitions.
2. In Section 3, we present the main results via neutrosophic e -compactness.
3. In Section 4, we contribute to the literature by constructing the concepts of NO (resp., NeO, $N \beta O$, $N \delta O$, $N \delta SO$, $N \delta PO$, and $NS e^* O$) and we study open map and continuity.
4. In Section 5, we introduce and study to the literature by constructing the concepts of NeO and we study neutrosophic locally e -compact (NL e -compact).

5. In Section 6, we explore and characterize some novel covering properties via neutrosophic e -base ($N e$ -base).
6. Finally, we wrap up the paper by presenting our conclusions and suggesting topics for future research.

2 | Preliminaries

Recalling the fundamental ideas and symbols that will be utilized throughout the article is the goal of this part. We start by providing a quick introduction to NSs and related structures, which introduce three separate components: truth membership, indeterminacy membership, and falsity membership. These structures generalize classical and fuzzy set frameworks. Building on this foundation, we present the relevant topological notions and their neutrosophic extensions, which serve as the groundwork for the results developed in later sections.

Definition 1 (see [28]). Classical topology, often referred to as general topology, constitutes a vital area of mathematics that examines the characteristics of spaces maintained under continuous transformations. A fundamental concept within topology is the topological space, which extends the idea of open sets found in metric spaces. Formally, a topological space is defined as a pair $(\mathcal{T}, \mathcal{U})$, where $\mathcal{U}$ is a nonempty set and $\mathcal{T}$ is a collection of subsets of $\mathcal{U}$ (called the topology) satisfying three conditions:

1. $\phi, X \in \mathcal{T}$.
2. $G_1 \cap G_2 \in \mathcal{T}$ for every $G_1, G_2 \in \mathcal{T}$.
3. $\cup G_i \in \mathcal{T}, \forall \{G_i | i \in I\} \subseteq \mathcal{T}$.

Definition 2 (see [24]). Consider a nonempty set represented by $\mathcal{U}$. A NS, referred to as (NS), is defined as $\tilde{Q} = \left\{ \langle r, \mu_{\tilde{Q}}(r), \sigma_{\tilde{Q}}(r), \gamma_{\tilde{Q}}(r) \rangle : r \in \mathcal{U} \right\}$, where $\gamma_{\tilde{Q}}(r)$ denotes the degree of non-membership, $\sigma_{\tilde{Q}}(r)$ represents the degree of indeterminacy, and $\mu_{\tilde{Q}}(r)$ indicates the degree of membership for each element $r \in \mathcal{U}$ within the set $\tilde{Q}$.

Definition 3 (see [24]). Consider the collection $\{Q_j : j \in J\}$ as an arbitrary family of NSS located in $\mathcal{U}$, then

1. Intersection and union $\cap Q_j, \cup Q_j$ may be defined as follows:

$$\begin{aligned} - T_1: \cap Q_j &= \left\langle u, \bigwedge_{j \in J} \mu_{Q_j}(u), \bigwedge_{j \in J} \sigma_{Q_j}(u), \bigvee_{j \in J} \gamma_{Q_j}(u) \right\rangle. \\ - T_2: \cap Q_j &= \left\langle u, \bigwedge_{j \in J} \mu_{Q_j}(u), \bigvee_{j \in J} \sigma_{Q_j}(u), \bigvee_{j \in J} \gamma_{Q_j}(u) \right\rangle. \\ - T_1: \cup Q_j &= \left\langle u, \bigvee_{j \in J} \mu_{Q_j}(u), \bigvee_{j \in J} \sigma_{Q_j}(u), \bigwedge_{j \in J} \gamma_{Q_j}(u) \right\rangle. \\ - T_2: \cup Q_j &= \left\langle u, \bigvee_{j \in J} \mu_{Q_j}(u), \bigwedge_{j \in J} \sigma_{Q_j}(u), \bigwedge_{j \in J} \gamma_{Q_j}(u) \right\rangle. \end{aligned}$$

Our primary objective is to create tools for the development of NSs and NT, so we need to present the NSs (NSs) 0_N and 1_N [24] within $\mathcal{U}$ as outlined in the following:

$2-1_N$ can be described as consisting of four different types:

1. $1_{Ne} = \{\langle u, 1, 1, 1 \rangle : u \in \mathcal{U}\}$,
2. $1_{Ne} = \{\langle u, 1, 1, 0 \rangle : u \in \mathcal{U}\}$,

$$3. 1_{Ne} = \{\langle u, 1, 0, 0 \rangle: u \in \mathcal{U}\},$$

$$4. 1_{Ne} = \{\langle u, 1, 0, 1 \rangle: u \in \mathcal{U}\}.$$

$1 - 0_N$ can be described as consisting of four different types:

$$1. 0_{Ne} = \{\langle u, 0, 1, 0 \rangle: u \in \mathcal{U}\},$$

$$2. 0_{Ne} = \{\langle u, 0, 0, 0 \rangle: u \in \mathcal{U}\},$$

$$3. 0_{Ne} = \{\langle u, 0, 0, 1 \rangle: u \in \mathcal{U}\},$$

$$4. 0_{Ne} = \{\langle u, 0, 1, 1 \rangle: u \in \mathcal{U}\}.$$

Definition 4 (see [29]). Assume $(\mathcal{U}, \mathcal{T})$ represents a neutrosophic topology space (NTS) curated on $\mathcal{U}$, while Q indicates a neutrosophic set (NS) on $\mathcal{U}$. The set Q qualifies as neutrosophic.

1. The concept of the neutrosophic interior for set Q , or $NInt(Q)$, is expressed as $NInt(Q) = \cup\{K: K \subseteq Q, K \text{ qualifies as a neutrosophic open set (NOS)} \in \mathcal{U}\}$.
2. The concept of the neutrosophic closure for set Q , or $NCl(Q)$, is expressed as $NCl(Q) = \cap\{L: Q \subseteq L, L \text{ is a NCS} \in \mathcal{U}\}$.

Definition 5 (see [15]). Let $\tilde{Q} = \langle \mu_{\tilde{Q}}(r), \sigma_{\tilde{Q}}(r), \gamma_{\tilde{Q}}(r) \rangle$ be an NS on $\mathcal{U}$. We can define the complement of the set $\tilde{Q}$, known concisely as $C(\tilde{Q})$, in the following way:

$$1. C(\tilde{Q}) = \left\{ \langle u, 1 - \mu_{\tilde{Q}}(u), 1 - \gamma_{\tilde{Q}}(u) \rangle: u \in \mathcal{U} \right\},$$

$$2. C(\tilde{Q}) = \left\{ \langle u, \gamma_{\tilde{Q}}(u), \sigma_{\tilde{Q}}(u), \mu_{\tilde{Q}}(u) \rangle: u \in \mathcal{U} \right\},$$

$$3. C(\tilde{Q}) = \left\{ \langle u, \gamma_{\tilde{Q}}(u), 1 - \sigma_{\tilde{Q}}(u), \mu_{\tilde{Q}}(u) \rangle: u \in \mathcal{U} \right\}.$$

Definition 6 (see [24]). Assume $\mathcal{U}$ is a nonempty set, with GNSS denoted by H and K defined as $H = \{r, \mu_H(r), \sigma_H(r), \gamma_H(r)\}$ and $K = \{u, \mu_K(u), \sigma_K(u), \gamma_K(u)\}$. We may evaluate two alternative definitions for the subset condition ($H \subseteq K$).

1. $T_2: Q \subseteq K \iff \mu_Q(u) \leq \mu_K(u), \sigma_Q(u) \geq \sigma_K(u), \text{ and } \gamma_Q(u) \geq \gamma_K(u).$
2. $T_1: Q \subseteq K \iff \mu_Q(u) \leq \mu_K(u), \sigma_Q(u) \leq \sigma_K(u), \text{ and } \gamma_Q(u) \geq \gamma_K(u).$

Definition 7 (see [24]). Consider $\tilde{Q} = \{\mu_{\tilde{Q}}(u), \sigma_{\tilde{Q}}(u), \gamma_{\tilde{Q}}(u)\}$ as a set of neutrosophic open (NO) sets, typically denoted by the acronym NROs. Also, assume $B = \{\mu_B(u), \sigma_B(u), \gamma_B(u)\}$ to be a NS on a NTS $(\mathcal{U}, \mathcal{T})$. Then,

1. $\tilde{Q}$ is called neutrosophic regular open iff $\tilde{Q} = NInt(NCl(\tilde{Q}))$.
2. The complement of NRO is neutrosophic regular closed (NRCs).

Definition 8 (see [30]). A family $\mathcal{T}$ of neutrosophic subsets forms a NT on a nonempty set $\mathcal{U}$, provided that it complies with the specified axioms.

1. $0_N, 1_N \in \mathcal{T}$.
2. $H_1 \wedge H_2 \in \mathcal{T}$ for any $H_1, H_2 \in \mathcal{T}$.
3. $\forall H_i \in \mathcal{T}, \forall \{H_i | i \in I\} \subseteq \mathcal{T}$.

The pair $(\mathcal{U}, \mathcal{T})$ is called a NTS.

Definition 9 (see [29]). We define $(\mathcal{U}, \mathcal{T})$ as a NTS on the set $\mathcal{U}$, where A represents a NS associated with $\mathcal{U}$. Here, we refer to a set A as neutrosophic.

1. The $N\delta Int(Q)$, or the δ -interior of Q , is characterized by the expression $N\delta Int(Q) = \forall\{K: K \subseteq Q, K \text{ qualifies as a NROs} \in \mathcal{U}\}$.
2. The concept of the δ -closure of Q , or $N\delta Cl(Q)$ for short, is articulated as

$$N\delta Cl(Q) = \wedge\{L: Q \subseteq L, L \text{ is a NRCs} \in \mathcal{U}\}. \quad (1)$$

Definition 10 (see [23]). In the framework of $(\mathcal{U}, \mathcal{T})$ considered as a nontopological space on $\mathcal{U}$, let A denote a non-singleton set. We define a set Q as follows:

A set Q qualifies as a neutrosophic e -open set, or NeOs for short, if it meets the requirement $Q \subseteq NCl(N\delta Int(Q)) \vee NInt(N\delta Cl(Q))$. In contrast, a neutrosophic e -closed set, similarly referred to as NeOs, is defined by $Q \supseteq NCl(N\delta Int(Q)) \wedge NInt(N\delta Cl(Q))$.

Definition 11 (see [29]). Assume $(\mathcal{U}, \mathcal{T})$ is a nontopological setting on $\mathcal{U}$, with A being a nonclosed entity within $\mathcal{U}$. In this framework, a set Q is referred to as follows:

A neutrosophic δ -open set (N δ Os) is established with the equation $Q = N\delta Int(Q)$. A neutrosophic δ -pre open set (N δ POs) is defined by $Q \subseteq NInt(N\delta Cl(Q))$. A N δ SOs exists if it satisfies $Q \subseteq NCl(N\delta Int(Q))$. The neutrosophic δ - α -open set, or α -open set (N $\delta \alpha$ Os or N α Os), is defined by the condition $Q \subseteq NInt(NCl(N\delta Int(Q)))$. For a neutrosophic e^* -open set (N e^* Os), it holds that $Q \subseteq NCl(NInt(N\delta Cl(Q)))$. Lastly, a neutrosophic β -open set (N β Os) is valid if $Q \subseteq NCl(NInt(NCl(Q)))$.

Remark 1 (see [15, 23]). From the aforementioned definition and the different kinds of NOS, the following diagram can be derived (see Figure 1).

Example 1. Let us define $\mathcal{U} = \{a, b, c\}$, where $Q = \langle (0.6, 0.6, 0.6), (0.4, 0.7, 0.7), (0.3, 0.6, 0.6) \rangle$, $K = \langle (0.2, 0.3, 0.3), (0.8, 0.5, 0.5), (0.2, 0.3, 0.3) \rangle$, and $F = \langle (0.5, 0.5, 0.5), (0.3, 0.6, 0.6), (0.6, 0.6, 0.6) \rangle$. Thus, $\mathcal{T} = \{0_N, 1_N, Q, K\}$ constitutes a NT on $\mathcal{U}$. The neutrosophic closed sets are specified as $F_1 = \langle (0.5, 0.5, 0.5), (0.6, 0.5, 0.5), (0.6, 0.5, 0.5) \rangle$ and $F_2 = \langle (0.8, 0.7, 0.7), (0.2, 0.6, 0.6), (0.8, 0.7, 0.7) \rangle$. Accordingly, the set F is regarded as a neutrosophic semiopen set (NSOs) though it does not qualify as a neutrosophic α -open set (N α Os).

Example 2. We define $\mathcal{U}$ as $\{a, b, c\}$, with $Q = \langle (0.6, 0.6, 0.6), (0.3, 0.6, 0.5), (0.3, 0.6, 0.5) \rangle$ and $K = \langle (0.2, 0.3, 0.3), (0.8, 0.6, 0.5), (0.3, 0.4, 0.4) \rangle$. Therefore, $\mathcal{T} = \{0_N, 1_N, Q, K\}$ acts as a NT on $\mathcal{U}$. The neutrosophic closed sets F_1 and F_2 can be specified as $F_1 = \langle (0.6, 0.6, 0.6), (0.7, 0.5, 0.5), (0.7, 0.5, 0.5) \rangle$ and $F_2 = \langle (0.8, 0.7, 0.7), (0.2, 0.5, 0.5), (0.8, 0.7, 0.7) \rangle$. Hence, Q serves as a NOS but is not classified as a neutrosophic regular open set (NROs), which means Q is a neutrosophic α -open set (N α Os).

Definition 12 (see [23]). Let $\tilde{Q}$ be an NS and $(\mathcal{U}, \mathcal{T})$ an NT, where $\tilde{Q} = \{r, \mu_{\tilde{Q}}(r), \sigma_{\tilde{Q}}(r), \gamma_{\tilde{Q}}(r)\}$. Then,

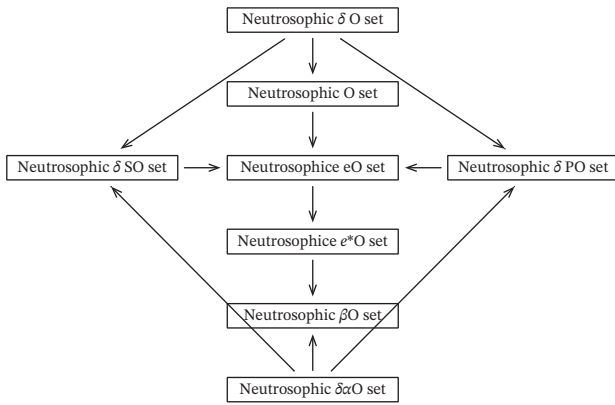


FIGURE 1 | Relation between neutrosophic open sets.

1. $NCL_e(\tilde{Q}) = \bigwedge \{H: H \text{ is a NeCS in } \mathcal{U} \text{ and } \tilde{Q} \subseteq H\}.$
2. $NInt_e(\tilde{Q}) = \bigwedge \{W: W \text{ is a NeOS in } \mathcal{U} \text{ and } W \subseteq \tilde{Q}\}.$

It can be understood that $\tilde{Q}$ functions as a NeCS (NeOS) in $\mathcal{U}$ precisely when $\tilde{Q}$ is equivalent to $CL_e(\tilde{Q})$ (and $\tilde{Q}$ is also equal to $Int_e(\tilde{Q})$).

Example 3. Consider the universal set $\mathcal{U} = \{a, b, c\}$ along with the neutrosophic structures $Q = \langle (0.3, 0.6, 0.3), (0.6, 0.6, 0.6), (0.3, 0.5, 0.3) \rangle$ and $K = \langle (0.8, 0.7, 0.6), (0.2, 0.3, 0.5), (0.2, 0.3, 0.3) \rangle$, as well as $D = \langle (0.6, 0.6, 0.6), (0.6, 0.6, 0.6), (0.6, 0.6, 0.6) \rangle$. This enables us to define a NT $\mathcal{T} = \{0_N, 1_N, Q, K\}$ on the universe $\mathcal{U}$. We then introduce two neutrosophic closed sets, denoted as F_1 and F_2 , characterized by $F_1 = \langle (0.7, 0.6, 0.7), (0.6, 0.6, 0.6), (0.7, 0.6, 0.6) \rangle$ and $F_2 = \langle (0.2, 0.3, 0.4), (0.8, 0.7, 0.6), (0.8, 0.7, 0.6) \rangle$. As a result, the set D qualifies as a neutrosophic preopen set (NPOs) but does not meet the criteria to be classified as a neutrosophic α -open set (N α Os).

Definition 13 (see [15]). Assume we have two NTSSs, $(\mathcal{U}, \mathcal{T}_1)$ and $(\mathcal{Z}, \mathcal{T}_2)$, and let $g: \mathcal{U} \rightarrow \mathcal{Z}$ be a specified function, then

1. Let us denote $\tilde{Q} = \{\mu_{\tilde{Q}}(Q), \sigma_{\tilde{Q}}(Q), \gamma_{\tilde{Q}}(Q)\}$ as a NS in the domain $\mathcal{Z}$. The preimage of $\tilde{Q}$ under the transformation g , which is written as $g^{-1}(\tilde{Q})$, forms a NS in $\mathcal{U}$. This is defined by the set $g^{-1}(\tilde{Q}) = \{g^{-1}(\mu_{\tilde{Q}}(Q)), g^{-1}(\sigma_{\tilde{Q}}(Q)), g^{-1}(\gamma_{\tilde{Q}}(Q))\}.$
2. Assume $\tilde{Q} = \{\mu_{\tilde{Q}}(Q), \sigma_{\tilde{Q}}(Q), \gamma_{\tilde{Q}}(Q)\}$ is a NS within the set $\mathcal{U}$; consequently, the output of $\tilde{Q}$ after applying the function g , noted as $g(\tilde{Q})$, becomes a NS in $\mathcal{Z}$, defined via the equation $g(\tilde{Q}) = \{g(\mu_{\tilde{Q}}(Q)), g(\sigma_{\tilde{Q}}(Q)), g(\gamma_{\tilde{Q}}(Q))\}.$

Definition 14 (see [15, 23]). Let $N(\mathcal{U})$ be the set of all NSs over $\mathcal{U}$. A NS $\tilde{Q} = \{r, \mu_{\tilde{Q}}(r), \sigma_{\tilde{Q}}(r), \gamma_{\tilde{Q}}(r): r \in \mathcal{U}\}$ is said to be a neutrosophic point (NP) if, for any element $z \in \mathcal{U}$, the membership value $\mu_{\tilde{Q}}(z)$ is equal to γ , the indeterminacy value $\sigma_{\tilde{Q}}(z)$ is equal to δ , and the nonmembership value $\gamma_{\tilde{Q}}(z)$ is equal to α when $z = r$. For all other elements $z \neq r$, the membership value $\mu_{\tilde{Q}}(z)$ is 0, the indeterminacy value $\sigma_{\tilde{Q}}(z)$ is 1, and the

nonmembership value $\gamma_{\tilde{Q}}(z)$ is 1. Here, $0 < \gamma \leq 1$, $0 \leq \delta < 1$, and $0 \leq \alpha < 1$. A NP $\tilde{Q} = \{r, \mu_{\tilde{Q}}(r), \sigma_{\tilde{Q}}(r), \gamma_{\tilde{Q}}(r): r \in \mathcal{U}\}$ will be denoted by $u_{\gamma, \alpha, \beta}$ and its complement denoted by $u_{\gamma, \alpha, \beta}^c$.

Definition 15 (see [31]). Denote $\mathcal{N}(U)$ as the complete set (i.e., all NSs) of NSs defined over U . A point in the neutrosophic space, $u_{\gamma, \alpha, \beta} \in \mathcal{N}(U)$, is termed quasicoincident with another NS $B \in \mathcal{N}(U)$, expressed as $u_{\gamma, \alpha, \beta} q B$, if either γ exceeds $\mu_B(u)$ or α is less than $\sigma_B(u)$.

A NS F is recognized as quasicoincident [15] with a NS Q at $x \in U$, or F quasicoincides with Q at $u \in U$, denoted as FqQ at u , if $\mu_F(u) > \mu_Q(u)$, $\sigma_F(u) < \sigma_Q(u)$, or $\gamma_F(u) < \gamma_Q(u)$ is true. For F to be classified as quasicoincident with Q , also written as FqQ , it requires that Q quasicoincides with itself at some point $u \in U$. Thus, F is quasicoincident with Q if there exists an element $x \in U$ such that either $\mu_F(u) > \mu_Q(u)$, $\sigma_F(u) < \sigma_Q(u)$, or $\gamma_F(u) < \gamma_Q(u)$ holds; in other words, $\gamma_F(u) > \gamma_Q(u)$, $\sigma_F(u) < 1_N - \sigma_Q(u)$, or $\gamma_F(u) < \mu_Q(u)$. If the NP $u_{\gamma, \alpha, \beta}$ is not quasicoincident with a NS F , we denote it by $u_{\gamma, \alpha, \beta} \hat{q} F$.

The example of an NS N qualifies as an ϵ -nbd of $u_{\gamma, \alpha, \beta}$ if there exists an NOS Q in the neighborhood $\mathcal{N}(U)$ for which $u_{\gamma, \alpha, \beta}$ is included in Q , and Q is not greater than N .

Similarly, if the NS F is not quasicoincident with the NS Q , we shall denote it by $F\hat{q}Q$. The set of all the points in U , at which FqQ , will be denoted by $F \mathfrak{U} A$, that is, $F \mathfrak{U} Q = \{x \in U: FqQ \text{ at } x\}$.

Definition 16 (see [26]). Consider $(\mathcal{U}, \mathcal{T})$ as a NTS. A set of neutrosophic closed subsets represented as $\{H_v: v \in \Lambda\}$ is said to exhibit the finite intersection property (FIP) if any finite subgroup $\{H_{v_i}: i = 1, 2, \dots, n\}$ drawn from $\{H_v: v \in \Lambda\}$ satisfies the condition $\bigcap_{i=1}^n H_{v_i} \neq \emptyset$, where Λ denotes the index set.

Definition 17 (see [26]). Within an NTS $(\mathcal{U}, \mathcal{T})$, a NS S is classified as a neutrosophic compact set, or N-compact, if all NOCs of S have a finite neutrosophic open subcover (NOSC). In addition, the space $\mathcal{U}$ is recognized as a neutrosophic compact space if every NOC associated with $\mathcal{U}$ exhibits a finite NOSC.

Definition 18 (see [15]). Consider $(\mathcal{U}, \mathcal{T})$ as a NTS, where $u_{\gamma, \alpha, \beta}$ signifies a NP in $\mathcal{U}$. A NS K is deemed a neutrosophic neighborhood (referred to as nhbd) in $N(\mathcal{U})$ if there exists a NS H in $\mathcal{T}$ that contains the NP $u_{\gamma, \alpha, \beta}$ and for which H is a subset of K .

Definition 19 (see [25]). In the context of an NTS $(\mathcal{U}, \mathcal{T})$, a NS S is identified as a neutrosophic compact set, or N-compact, if all of its NOCs demonstrate a finite NOSC. Moreover, the space $\mathcal{U}$ is classified as a neutrosophic compact space if every NOC associated with $\mathcal{U}$ shows a finite NOSC.

Assume G is a NeOC of the NS S , where G' is a subset of G . In this context, G' qualifies as a neutrosophic e -open subcover [25] (or NeOSC for short) of G if it sufficiently covers S .

A NO covering of set S is termed finite (or countable) if it comprises a finite (or countable) collection of neutrosophic e -open sets.

Definition 20 (see [26]). Consider $(U, \mathcal{T})$ as a NTS with S as an element of $N(U)$. A set of NOS $G = \{H_v: v \in \Lambda\}$ is defined as a neutrosophic open cover (NOC) for S if it satisfies the condition $S \subseteq \bigvee_{v \in \Lambda} H_v$. Thus, it can be said that G covers S . Generally, we refer to G as an NOC of U when S is equal to $\bigvee_{v \in \Lambda} H_v$.

Consider G as a NOC associated with the set S , where G' is contained within G . G' is recognized as a NOSC of G if it covers the whole set S . A NOC for S is termed finite (or countable) if it consists of a finite (or countable) collection of NOS.

3 | Neutrosophic e -Compactness

Definition 21 (see [25]). A NT $(U, \mathcal{T})$ is referred to as neutrosophic e -compact (or simply $N e$ -compact) if every NeOSs cover of U has a finite subcover. Moreover, $(U, \mathcal{T})$ is regarded as $N e$ -compact concerning U if every NeOSs cover of U includes a finite subcover.

Definition 22 (see [32]). A NT expressed as $(U, \mathcal{T})$ is characterized as $N \delta$ S-compact (resp., $N \delta$ P-compact, $N \delta$ - α -compact, $N e^*$ -compact, and $N \beta$ -compact) if every collection of $N \delta$ SOs (and likewise $N \delta$ POs, $N \delta \delta$ - α Os, $N e^*$ Os, and $N \beta$ Os) that covers U has a finite subcover available.

Definition 23. Let $(U, \mathcal{T})$ represent a nonstandard structure (NTS). A set of neutrosophic collections $\{D_\alpha: \alpha \in \Delta\}$ is considered to exhibit the FIP if every finite subcollection $\{D_{\alpha_i}: i = 1, 2, 3, \dots, n\}$ from $\{D_\alpha: \alpha \in \Delta\}$ meets the requirement that $\bigcap_{i=1}^n D_{\alpha_i} \neq \phi$, where Δ serves as an indexing set.

A topology $(U, \mathcal{T})$ is defined as a neutrosophic e -Lindelöf space (or neutrosophic countably e -compact space) if and only if each NeO-compact (including countably NeO-compact) set within U has a NeOSC that is either countable or finite.

Proposition 1. All neutrosophic e -compact spaces are also considered neutrosophic countably e -compact spaces.

Proof 1. The proof is obvious. $\square$

Lemma 1. In NT $(U, \mathcal{T})$, each neutrosophic e -compact space is a neutrosophic compact space.

Proof 2. The proof is direct from Theorem 1. $\square$

In the following example, the notions C_N and I_N are known as closure and interior in NT.

Example 4. Consider the NTS $(U, \mathcal{T})$, where $U = \{x, y\}$, then

$$\begin{aligned} H_1 &= \left\langle z, \left(\frac{x}{0.3}, \frac{y}{0.2}\right), \left(\frac{x}{0.5}, \frac{y}{0.5}\right), \left(\frac{x}{0.4}, \frac{y}{0.35}\right) \right\rangle, \\ H_2 &= \left\langle z, \left(\frac{x}{0.3}, \frac{y}{0.5}\right), \left(\frac{x}{0.7}, \frac{y}{0.2}\right), \left(\frac{x}{0.4}, \frac{y}{0.35}\right) \right\rangle. \end{aligned} \quad (2)$$

Then, the family $\mathcal{Y} = \{0, 1, H_1\}$ is an NT on U . Since $H_2 \leq C_N(I_N(\delta C_N(H_2))) = \overline{H_1}$ and $H_2 \leq C_N(I_N(\delta C_N(H_2))) = \overline{H_1}$, then H_2 is an $N \delta \beta$ OS in U but not NeOS since $H_2 \not\leq C_N(\delta I_N(H_2)) \vee I_N(\delta C_N(H_2)) = 0 \vee H_1 = H_1$.

Example 5. Consider the NTS $(U, \mathcal{T})$, where $U = \{x, y\}$, then

$$\begin{aligned} H_1 &= \left\langle z, \left(\frac{x}{0.2}, \frac{y}{0.1}\right), \left(\frac{x}{0.5}, \frac{y}{0.5}\right), \left(\frac{x}{0.4}, \frac{y}{0.35}\right) \right\rangle, \\ H_2 &= \left\langle z, \left(\frac{x}{0.3}, \frac{y}{0.5}\right), \left(\frac{x}{0.7}, \frac{y}{0.2}\right), \left(\frac{x}{0.4}, \frac{y}{0.35}\right) \right\rangle. \end{aligned} \quad (3)$$

Then, the family $\mathcal{Y} = \{0, 1, H_1\}$ is an NT on U . Since $H_2 \leq C_N(\delta I_N(H_2)) \vee I_N(\delta C_N(H_2)) = \overline{H_1} \vee H_1 = \overline{H_1}$, H_2 is a NeOS, and $N \delta \beta$ OS but not $N \delta$ POS, and NaOS; hence, $H_2 \not\leq I_N(\delta C_N(H_2)) = H_1$ and $H_2 \not\leq I_N(C_N(\delta I_N(H_2))) = H_1$.

Example 6. Consider the NTS $(U, \mathcal{T})$, where $U = \{x, y, d, e\}$, then

$$\begin{aligned} H_1 &= \left\langle z, \left(\frac{x}{1}, \frac{y}{0}, \frac{d}{0.2}, \frac{e}{0.1}\right), \left(\frac{x}{0}, \frac{y}{1}, \frac{d}{0.7}, \frac{e}{1}\right), \left(\frac{x}{1}, \frac{y}{0.3}, \frac{d}{1}, \frac{e}{0.1}\right) \right\rangle, \\ H_2 &= \left\langle z, \left(\frac{x}{0}, \frac{y}{1}, \frac{d}{0}, \frac{e}{0}\right), \left(\frac{x}{1}, \frac{y}{0}, \frac{d}{1}, \frac{e}{0.1}\right), \left(\frac{x}{1}, \frac{y}{0.3}, \frac{d}{0}, \frac{e}{0.1}\right) \right\rangle, \\ H_3 &= \left\langle z, \left(\frac{x}{1}, \frac{y}{0}, \frac{d}{0}, \frac{e}{1}\right), \left(\frac{x}{0}, \frac{y}{0.2}, \frac{d}{0}, \frac{e}{0}\right), \left(\frac{x}{0}, \frac{y}{0.2}, \frac{d}{0}, \frac{e}{0.1}\right) \right\rangle. \end{aligned} \quad (4)$$

Then, the family $\mathcal{Y} = \{0, 1, H_1, H_2, H_1 \vee H_2\}$ is an NT on U . Since $H_3 \leq C_N(\delta I_N(H_2)) \vee I_N(\delta C_N(H_2)) = 0 \vee 1 = 1$, H_3 is a NeOS, but not $N \delta$ SOS, and NaOS; hence, $H_3 \not\leq C_N(\delta I_N(H_3)) = 0$.

Example 7. Consider the NTS $(U, \mathcal{T})$, where $U = \{a, b\}$, then

$$K_n = \left\langle z, \left(\frac{a}{(n/(n+1))}, \frac{b}{((n+2)/(n+1))}\right), \left(\frac{a}{(1/(n+3))}, \frac{b}{(1/(n+4))}\right), \left(\frac{a}{(1/(n+3))}, \frac{b}{(1/(n+3))}\right) \right\rangle. \quad (5)$$

The collection $\mathcal{T} = \{0, 1\} \cup \{K_n: n \in \mathbb{N}\}$ is defined. It is important to observe that $\bigcup_{n \in \mathbb{N}} K_n$ serves as an open cover for U ; however, this cover cannot be reduced to a finite subcover. Consider the following equations:

$$\begin{aligned} K_1 &= \left\langle z, \left(\frac{a}{0.6}, \frac{b}{0.7}\right), \left(\frac{a}{0.30}, \frac{b}{0.4}\right), \left(\frac{a}{0.5}, \frac{b}{0.40}\right) \right\rangle, \\ K_2 &= \left\langle z, \left(\frac{a}{0.80}, \frac{b}{0.7}\right), \left(\frac{a}{0.3}, \frac{b}{0.30}\right), \left(\frac{a}{0.5}, \frac{b}{0.40}\right) \right\rangle, \end{aligned}$$

$$K_3 = \left\langle z, \left(\frac{a}{0.9}, \frac{b}{0.80}\right), \left(\frac{a}{0.3}, \frac{b}{0.17}\right), \left(\frac{a}{0.3}, \frac{b}{0.30}\right) \right\rangle, \quad (6)$$

and observe that $K_1 \cup K_2 \cup K_3$. So, for any finite subcollection, $\{K_{n_j}: j \in \Lambda, \text{ where } \Lambda \text{ is a finite subset of } \mathbb{N}\}$, $\bigcup_{n_j \in \Lambda} K_{n_j} = K_w = 1_N$, where $w = \max\{n_j: n_j \in \Lambda\}$. Therefore, NTS $(U, \mathcal{T})$ is not compact.

Theorem 1. In NT $(\mathcal{U}, \mathcal{T})$, every neutrosophic e -compact set is a neutrosophic compact set.

Proof 3. Suppose H is a neutrosophic e -compact set of an NTS $(\mathcal{U}, \mathcal{T})$. Let $K = \{D_\alpha: \alpha \in \Lambda\}$ be an NOC of B . Hence, each D_α is an NEO set for each $\alpha \in \Lambda$. Then, K is an NeO-cover of B . Hence, H is e -compact, so there exists a finite subcollection $\{D_j^1, D_j^2, D_j^3, \dots, D_j^n\}$, but K such that $H \subseteq D_j^1 \cup D_j^2 \cup D_j^3 \cup \dots \cup D_j^n$. So, the NOC K of H has a finite NOSC $\{D_j^1, D_j^2, D_j^3, \dots, D_j^n\}$. Then, H is a neutrosophic compact set. $\square$

Proposition 2. Within an NTS, when two neutrosophic e -compact sets are united, their union remains neutrosophic e -compact.

Proof 4. Let H and K represent two neutrosophic e -compact sets within an NTS $(\mathcal{U}, \mathcal{T})$. Define $J = \{D_\alpha: \alpha \in \Lambda\}$ as an NEOC of the union of H and K . Consequently, we have $H \cup K \subseteq \cup_{\alpha \in \Lambda} D_\alpha$. Since H is a subset of $H \cup K$, it follows that J serves as an NEOC for H . Additionally, acknowledging that H is neutrosophic e -compact indicates the existence of a finite subcollection $\{D_j^1, D_j^2, D_j^3, \dots, D_j^n\}$ drawn from J such that H is contained within these sets. Likewise, because K is also neutrosophic e -compact, a finite subcollection $\{B_j^1, B_j^2, B_j^3, \dots, B_j^n\}$ can be found in J where K is included. Therefore, we conclude that $H \cup K$ is a subset of $D_j^1, D_j^2, D_j^3, \dots, D_j^n \cup B_j^1, B_j^2, B_j^3, \dots, B_j^n$. This implies that a finite subcollection $\{D_j^1, D_j^2, D_j^3, \dots, D_j^n, B_j^1, B_j^2, B_j^3, \dots, B_j^n\}$ exists within J ensuring that $H \cup K \subseteq D_j^1, D_j^2, D_j^3, \dots, D_j^n \cup B_j^1, B_j^2, B_j^3, \dots, B_j^n$. Thus, we ascertain that $H \cup K$ is neutrosophic e -compact. This proves our assertion. $\square$

Example 8. The opposite of Theorem 1 does not hold true. We demonstrate this with the following example.

Consider $\mathcal{U} = \{x, y\}$, $Q = \{\langle x, 0, 1, 1 \rangle, \langle y, 1, 0, 0 \rangle\}$, and $R_m = \{\langle x, 0, 1, 1 \rangle, \langle y, (m/(m+1)), (1/m), 1/(m+1) \rangle\}$, where $m \in N = \{1, 2, 3, \dots\}$ and $\mathcal{T} = \{\mathcal{U}, \phi, Q\}$. It is clear that $(\mathcal{U}, \mathcal{T})$ forms a NT, with R_m acting as an NEO set for each integer $m \in N$. Moreover, Q is clearly identified as a neutrosophic compact set. It is worth noting that $\{R_m: m \in N\}$ serves as a NeO-cover for Q although it does not possess a NeOSC. This indicates that Q is not a neutrosophic e -compact set.

Based on the definitions of compactness outlined above, the relationships shown in the diagram are evident.

Remark 2 (see [32]). From the above definition and some types of NOSs, we have the following diagram (see Figure 2).

Proposition 3. In the context of a NTS, the union of a neutrosophic e -compact set and a neutrosophic compact set forms a neutrosophic compact set.

Proof 5. This is obvious. $\square$

Proposition 4. Within an NTS framework, any finite collection of neutrosophic e -compact sets is still classified as neutrosophic e -compact.

Proof 6. From the moment Proposition 2 is made. $\square$

Definition 24. Consider $(\mathcal{V}, \mathcal{T}|_{\mathcal{V}})$ functioning as a neutrosophic dimension within the NTS $(\mathcal{U}, \mathcal{T})$. We denote by $NEO(\mathcal{V})$ the set of NEO sets $D|_{\mathcal{V}}$ within $\mathcal{V}$ for which D is recognized as a NEO set in $\mathcal{U}$. Specifically, $NEO(\mathcal{V})$ is defined as $\{D|_{\mathcal{V}} \subseteq \mathcal{V}: D|_{\mathcal{V}}\}$ being a NEO set in $\mathcal{V}$ and $D \subseteq \mathcal{U}$ being a NEO set in $\mathcal{U}$.

Proposition 5. Consider $(\mathcal{V}, \mathcal{T}|_{\mathcal{V}})$ as a neutrosophic subspace within the NTS $(\mathcal{U}, \mathcal{T})$ and let $H \subseteq \mathcal{V}$. The set H qualifies as neutrosophic e -compact in $\mathcal{U}$ if every covering of H by subsets from $NEO(\mathcal{V})$ contains a finite subcover.

Proof 7. We define $J = \{D_\alpha|_{\mathcal{V}}: \alpha \in \Lambda\}$ as a covering of H , where for all $\alpha \in \Lambda$, $D_\alpha|_{\mathcal{V}}$ is included in $NEO(\mathcal{V})$. Therefore, if $H \subseteq \cup_{\alpha \in \Lambda} D_\alpha|_{\mathcal{V}}$, this leads to $H \subseteq \cup_{\alpha \in \Lambda} D_\alpha$. Clearly, each D_α is a NEO set in $\mathcal{U}$, as stated in Definition 24, for all $D_\alpha|_{\mathcal{V}}$ in the collection J . Consequently, we find that $J^* = \{D_\alpha: D_\alpha|_{\mathcal{V}} \in NEO(\mathcal{V})\}$ serves as an NEOC for H within $\mathcal{U}$. Since H is e -compact in $\mathcal{U}$, there must be a finite subset $\{D_{ai}: i = 1, 2, 3, \dots, n\}$ from J^* where $H \subseteq (\cup_{i=1}^n D_{ai})$ leads to $H \subseteq \cup_{i=1}^n (D_{ai}|_{\mathcal{V}})$. Hence, the cover J for H admits a finite subcover $\{D_{ai}: i = 1, 2, 3, \dots, n\}$.

Sufficiently, let $K = \{D_\alpha: \alpha \in \Lambda\}$ be an NEOC of H in $\mathcal{U}$, where D_α is an NEO set in $\mathcal{U}$ for each $\alpha \in \Lambda$. Then, $H \subseteq \cup_{\alpha \in \Lambda} D_\alpha \implies H \subseteq (\cup_{\alpha \in \Lambda} D_\alpha)|_{\mathcal{V}} \implies H \subseteq \cup_{\alpha \in \Lambda} D_\alpha(D_\alpha|_{\mathcal{V}})$. Since $D_\alpha|_{\mathcal{V}} \in NEO(\mathcal{V})$ for each $D_\alpha \in K$ by Definition 23, so $K^* = \{D_\alpha|_{\mathcal{V}}: \alpha \in \Lambda\}$ is a collection H covers the set H through the NEO constructs present in $NEO(\mathcal{V})$. Therefore, as per the premise, a finite subcollection $\{D_{ai}|_{\mathcal{V}}: i = 1, 2, 3, \dots, n\}$ from K^* exists such that H falls within the union $\cup_{i=1}^n D_{ai}|_{\mathcal{V}}$. This leads to the conclusion that H is also included in $(\cup_{i=1}^n D_{ai})|_{\mathcal{V}}$, thereby confirming that $H \subseteq \cup_{i=1}^n D_{ai}$. Consequently, the NEOC K for H possesses a finite NEOSC, represented by $\{D_{ai}|_{\mathcal{V}}: i = 1, 2, 3, \dots, n\}$. Therefore, it follows that H is neutrosophic e -compact in $\mathcal{U}$. $\square$

Proposition 6. Let $(\mathcal{V}, \mathcal{T}|_{\mathcal{V}})$ represent a neutrosophic subspace of the NTS $(\mathcal{U}, \mathcal{T})$ and define $H \subseteq \mathcal{V}$. If H qualifies as neutrosophic e -compact within $\mathcal{U}$, it follows that H is also neutrosophic compact in $\mathcal{V}$.

Proof 8. Define $J = \{D_\alpha|_{\mathcal{V}}: \alpha \in \Lambda\}$ as an NOC of H within $\mathcal{V}$, where $D_\alpha|_{\mathcal{V}}$ belongs to $\mathcal{T}|_{\mathcal{V}}$ for each $\alpha \in \Lambda$. Thus, if $H \subseteq \cup_{\alpha \in \Lambda} (D_\alpha|_{\mathcal{V}})$, it implies that $H \subseteq \cup_{\alpha \in \Lambda} D_\alpha$. It is clear that D_α is an element of $\mathcal{T}$, indicating that each D_α is a NEO set within $\mathcal{U}$. Consequently, we can form $J^* = \{D_\alpha: D_\alpha|_{\mathcal{V}} \in J\}$, which serves as a NEOC of H . Since H is e -compact in $\mathcal{U}$, there exists a finite subcollection $\{D_{ai}|_{\mathcal{V}}: i = 1, 2, 3, \dots, n\}$ from J^* such that $H \subseteq \cup_{i=1}^n D_{ai}$, leading to $H \subseteq (\cup_{i=1}^n D_{ai})|_{\mathcal{V}}$, and thus $H \subseteq (\cup_{i=1}^n (D_{ai}))|_{\mathcal{V}}$. In this way, the NOC J associated with H includes a finite NOSC expressed as $\{(D_{ai})|_{\mathcal{V}}: i = 1, 2, 3, \dots, n\}$. Hence, H is recognized as neutrosophic compact in the space $\mathcal{V}$. $\square$

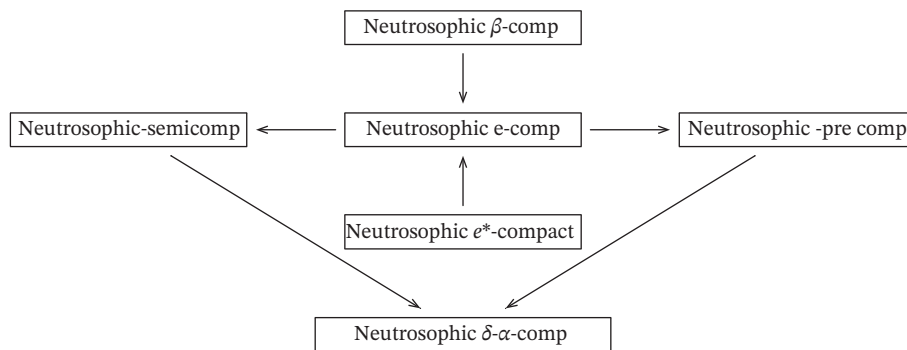


FIGURE 2 | Relation between neutrosophic compactness.

Theorem 2. When D qualifies as a NEC subset within a neutrosophic e -compact space $(\mathcal{U}, \mathcal{T})$, and D has no intersection with its complement D^c , it follows that D is neutrosophic compact.

Proof 9. This follows immediately from Theorem 2 since e -compactness implies compactness. $\square$

Theorem 3. In a neutrosophic e -compact space $(\mathcal{U}, \mathcal{T})$, if D is a NEC subset such that D and its complement D^c have no elements in common, then D is considered neutrosophic e -compact.

Proof 10. Let $J = \{H_\alpha: \alpha \in \Lambda\}$ represent a NEOC associated with D . Consequently, the set A is contained within the union of all H_α , where α belongs to Λ . Given that D^c is a NEO set and that D and D^c are disjoint, meaning $D \cup D^c = \mathcal{U}$, it follows that $Q = \{H_\alpha: \alpha \in \Lambda\} \cup \{D^c\}$ forms a NEOC of $\mathcal{U}$. As $\mathcal{U}$ is neutrosophic e -compact, there exists a finite subcollection $Q' = \{H_{\alpha_1}, H_{\alpha_2}, \dots, H_{\alpha_n}\} \cup D^c$ from Q such that $\mathcal{U}$ is included in the union $H_{\alpha_1} \cup H_{\alpha_2} \cup \dots \cup H_{\alpha_n} \cup D^c$. Therefore, we have $D \subseteq H_{\alpha_1} \cup H_{\alpha_2} \cup \dots \cup H_{\alpha_n} \cup D^c$. Since D and D^c are disjoint, it leads to $D \subseteq H_{\alpha_1} \cup H_{\alpha_2} \cup \dots \cup H_{\alpha_n}$. This indicates that the NEOC J pertaining to D contains a finite NEOSC represented by $\{H_{\alpha_1} \cup H_{\alpha_2} \cup \dots \cup H_{\alpha_n}\}$. Thus, D qualifies as a neutrosophic e -compact set. $\square$

Corollary 1. If D is a neutrosophic closed subset within a neutrosophic e -compact space $(\mathcal{U}, \mathcal{T})$ and the intersection of D and its complement D^c is empty, then D can be considered neutrosophic compact.

Proof 11. This is a direct result of Theorem 4. $\square$

Theorem 4. When D is a neutrosophic closed subset of a neutrosophic e -compact space $(\mathcal{U}, \mathcal{T})$ such that $D \cap D^c = \emptyset$, it follows that D is also neutrosophic e -compact.

Proof 12. As every neutrosophic closed set is a NEC set, this follows immediately from Theorem 2. $\square$

Theorem 5. Let $(\mathcal{U}, \mathcal{T})$ denote a NTS. A nonsingular structure $H = \{\langle z, \mu_H(z), \sigma_H(z), \gamma_H(z) \rangle: z \in \mathcal{U}\}$ defined on $\mathcal{U}$ is neutrosophic e -compact if for any collection $J = \{H_\alpha: \alpha \in \Lambda\}$ of NEO sets from $\mathcal{U}$ that meets the criteria $\mu_H(z) \leq \bigvee_{\alpha \in \Lambda} \mu_{D_\alpha}(z)$, $1 - \sigma_H(z) \leq \bigvee_{\alpha \in \Lambda} (1 - \sigma_{D_\alpha}(z))$, and $1 - \gamma_H(z) \leq \bigvee_{\alpha \in \Lambda} (1 - \gamma_{D_\alpha}(z))$, there exists a finite subset $\{D_{\alpha_i}: i = 1, 2, 3, \dots, n\}$ such that $\mu_H(z) \leq \bigvee_{i=1}^n \mu_{D_{\alpha_i}}(z)$, $1 - \sigma_H(z) \leq \bigvee_{i=1}^n (1 - \sigma_{D_{\alpha_i}}(z))$, and $1 - \gamma_H(z) \leq \bigvee_{i=1}^n (1 - \gamma_{D_{\alpha_i}}(z))$.

Proof 13. Let $J = \{H_\alpha: \alpha \in \Lambda\}$ represent a collection of NEO sets within $\mathcal{U}$ that meet the conditions $\mu_H(z) \leq \bigvee_{\alpha \in \Lambda} \mu_{D_\alpha}(z)$, $1 - \sigma_H(z) \leq \bigvee_{\alpha \in \Lambda} (1 - \sigma_{D_\alpha}(z))$, and $1 - \gamma_H(z) \leq \bigvee_{\alpha \in \Lambda} (1 - \gamma_{D_\alpha}(z))$.

Given that $1 - \sigma_H(z) \leq \bigvee_{\alpha \in \Lambda} (1 - \sigma_{D_\alpha}(z))$, it follows that $1 - \sigma_H(z) \leq 1 - \sigma_{D_\beta}(z)$ for some $\beta \in \Lambda$. This leads to $\sigma_H(z) \geq \sigma_{D_\beta}(z)$, implying $\sigma_H(z) \geq \bigvee_{\alpha \in \Lambda} \sigma_{D_\alpha}(z)$. In the same manner, we have $1 - \gamma_H(z) \leq \bigvee_{\alpha \in \Lambda} (1 - \gamma_{D_\alpha}(z))$, resulting in $\gamma_H(z) \geq \bigvee_{\alpha \in \Lambda} \gamma_{D_\alpha}(z)$. Hence, H is included in $\bigcup_{\alpha \in \Lambda} D_\alpha$, meaning J is a NEOC of H . Considering that H is neutrosophic e -compact, it follows that J contains a finite NEOSC $\{D_{\alpha_i}: i = 1, 2, 3, \dots, n\}$. Therefore, we conclude that $H \subseteq \bigcup_{i=1}^n D_{\alpha_i}$. The relationships $\mu_H(z) \leq \bigwedge_{i=1}^n \mu_{D_{\alpha_i}}(z)$, $\sigma_H(z) \geq \bigvee_{i=1}^n \sigma_{D_{\alpha_i}}(z)$, and $\gamma_H(z) \geq \bigwedge_{i=1}^n \gamma_{D_{\alpha_i}}(z)$ hold true. Now, since $\sigma_H(z) \geq \bigwedge_{i=1}^n \sigma_{D_{\alpha_i}}(z)$, we know that for some j , $1 \leq j \leq n$, $\sigma_{D_{\alpha_j}}(z) \implies \sigma_D(z) \geq \sigma_{D_{\alpha_j}}(z)$. This implies $1 - \sigma_D(z) \leq 1 - \sigma_{D_{\alpha_j}}(z)$ for the same j , which leads to $1 - \sigma_D(z) \leq \bigvee_{i=1}^n (1 - \sigma_{D_{\alpha_i}}(z))$. In parallel, we observe that $\gamma_H(z) \geq \bigwedge_{i=1}^n (1 - \gamma_{D_{\alpha_i}}(z))$. Consequently, we arrive at $\mu_H(z) \leq \bigvee_{i=1}^n \mu_{D_{\alpha_i}}(z)$, leading to $1 - \mu_D(z) \leq \bigvee_{i=1}^n \mu_{D_{\alpha_i}}(z)$, which further implies $1 - \bigvee_{i=1}^n \sigma_D(z) \leq \bigvee_{i=1}^n (1 - \sigma_{D_{\alpha_i}}(z))$, along with $1 - \gamma_D(z) \leq \bigvee_{i=1}^n (1 - \gamma_{D_{\alpha_i}}(z))$.

Let $J = \{H_\alpha: \alpha \in \Lambda\}$ denote a NEOC of H . Then, it follows that H is contained within the union of all D_α for $\alpha \in \Lambda$, or in other words, $\mu_H(z) \leq \bigvee_{\alpha \in \Lambda} \mu_{D_\alpha}(z)$, $\sigma_H(z) \geq \bigwedge_{\alpha \in \Lambda} \sigma_{D_\alpha}(z)$, and $\gamma_H(z) \geq \bigwedge_{\alpha \in \Lambda} \gamma_{D_\alpha}(z)$. Given that $\sigma_H(z) \geq \bigwedge_{\alpha \in \Lambda} \sigma_{D_\alpha}(z)$, it indicates that for some $\alpha \in H$, $\sigma_H(z) \geq \sigma_{D_\alpha}(z)$ and consequently, $1 - \sigma_H(z) \leq 1 - \sigma_{D_\alpha}(z)$, leading to the conclusion that $1 - \sigma_H(z) \leq \bigvee_{\alpha \in \Lambda} (1 - \sigma_{D_\alpha}(z))$. In a similar manner, considering $\gamma_H(z) \geq \bigwedge_{\alpha \in \Lambda} \gamma_{D_\alpha}(z)$ results in $1 - \gamma_H(z) \leq \bigvee_{\alpha \in \Lambda} (1 - \gamma_{D_\alpha}(z))$. Therefore, the collection J confirms the conditions $\mu_H(z) \leq \bigvee_{\alpha \in \Lambda} \mu_{D_\alpha}(z)$, $1 - \sigma_H(z) \leq \bigvee_{\alpha \in \Lambda} (1 - \sigma_{D_\alpha}(z))$, and $1 - \gamma_H(z) \leq \bigvee_{\alpha \in \Lambda} (1 - \gamma_{D_\alpha}(z))$. By assumption, there exists a finite subcollection $\{D_{\alpha_i}: i = 1, 2, 3, \dots, n\}$ satisfying $\mu_H(z) \leq \bigvee_{i=1}^n \mu_{D_{\alpha_i}}(z)$, $1 - \sigma_H(z) \leq \bigvee_{i=1}^n (1 - \sigma_{D_{\alpha_i}}(z))$, and $1 - \gamma_H(z) \leq \bigvee_{i=1}^n (1 - \gamma_{D_{\alpha_i}}(z))$. Consequently, from $1 - \sigma_H(z) \leq \bigvee_{i=1}^n (1 - \sigma_{D_{\alpha_i}}(z))$, it follows that for some j , $1 \leq j \leq n$, we have $1 - \sigma_H(z) \leq 1 - \sigma_{D_{\alpha_j}}(z)$, which implies $\sigma_H(z) \geq \sigma_{D_{\alpha_j}}(z)$, and hence $\sigma_H(z) \geq \bigwedge_{i=1}^n \sigma_{D_{\alpha_i}}(z)$. Likewise, one can ascertain that $\mu_H(z) \geq \bigwedge_{i=1}^n \gamma_{D_{\alpha_i}}(z)$. Thus, H is contained within the union of D_{α_i} for $i = 1$ to n , indicating that the NEOC J of H has a finite NEOSC $\{D_{\alpha_i}: i = 1, 2, 3, \dots, n\}$. Consequently, H is classified as a neutrosophic e -compact set. This concludes the proof. $\square$

4 | Neutrosophic e -Compactness and Functions

In this section, we contribute to the literature by constructing the concepts of NO (resp., $N\delta PO$, $N\delta O$, $N\delta SO$, NeO , $NS e^* O$, and $N\beta O$) and we study open map, continuity, and irresoluteness. Then, we study the relation between all the concepts and there prosperities.

Definition 25 (see [11]). Consider two NTSS, $(\mathcal{U}, \mathcal{T}_1)$ and $(\mathcal{V}, \mathcal{T}_2)$. A mapping $g: \mathcal{U} \rightarrow \mathcal{V}$ qualifies as a NO map, or as δS , δ , δP , e , e^* , and β open functions (N δ SO, N δ O, N δ PO, NeO, NS e^* O, and N β O), if for each NSOs in $(\mathcal{U}, \mathcal{T}_1)$, its image is a NOS (or N δ SOs, NOS, N δ POs, NeOs, N e^* Os, and N β Os) in $(\mathcal{V}, \mathcal{T}_2)$.

Remark 3. From the above definition and some types of NOS's, we have the following diagram (see Figure 3):

Definition 26. Consider two NTSS, $(\mathcal{U}, \mathcal{T}_1)$ and $(\mathcal{V}, \mathcal{T}_2)$, along with a function $g: \mathcal{U} \rightarrow \mathcal{V}$. We define g as being called

1. A function is considered neutrosophic e_* -continuous if the preimage $f^{-1}(G)$ corresponds to a NeO set in $\mathcal{U}$ for every NeO set G within $\mathcal{V}$.
2. Neutrosophic continuous [23] (N-conts.) occurs if and only if for every NOS in $\mathcal{V}$, its preimage also qualifies as a NOS in $\mathcal{U}$.
3. A neutrosophic e_* -open function occurs when the image $f(G)$ is classified as a NeO set in $\mathcal{V}$ for each NeO set G from $\mathcal{U}$.
4. Neutrosophic e -continuous [23] (N e -conts.) is defined such that the preimage of each NOS in $\mathcal{V}$ corresponds to a NeOS in $\mathcal{U}$.
5. Neutrosophic e -irresolute [23] (concise version N e -irrl.) holds true if and only if the preimage of every NeOS in $\mathcal{V}$ remains a NeOS in $\mathcal{U}$.
6. Neutrosophic open [33] (or e -open, noted as NO and NeO, respectively) is characterized by the condition that the image of each NOS in $\mathcal{V}$ transforms into a NOS (or NeOS) in $\mathcal{U}$.

Definition 27. Consider two NTSS, denoted as $(\mathcal{U}, \mathcal{T}_1)$ and $(\mathcal{V}, \mathcal{T}_2)$, along with a function $g: \mathcal{U} \rightarrow \mathcal{V}$. The function g is referred to as preneutrosophic e -open (pre-NeO) if the image of every NeOS in $\mathcal{V}$ corresponds to a NeOS in $\mathcal{U}$. Additionally, a function is defined as N e^* -irrl.) when the preimage of every NeOS in $\mathcal{V}$ is a NeOS in $\mathcal{U}$.

Remark 4 (see [32]). Given the definitions mentioned above, the implications depicted in the next two diagrams are irreversible (see Figure 4).

Example 9. Let $\mathcal{U} = \mathcal{Z} = \{a, b, c\}$, with $Q = \langle (0.3, 0.6, 0.3), (0.6, 0.6, 0.6), (0.3, 0.6, 0.3) \rangle$, $K = \langle (0.8, 0.7, 0.6), (0.2, 0.3, 0.6), (0.2, 0.3, 0.4) \rangle$, and $D = \langle (0.6, 0.6, 0.6), (0.6, 0.6, 0.6), (0.6, 0.6, 0.6) \rangle$. Therefore, $\mathcal{T}_1 = \{0_N, 1_N, Q, K\}$ is identified as a NT on $\mathcal{U}$, while $\mathcal{T}_2 = \{0_N, 1_N, D\}$ serves as a NT on $\mathcal{Z}$. It follows that $(\mathcal{U}, \mathcal{T}_1)$ and $(\mathcal{Z}, \mathcal{T}_2)$ represent NTSS. Define the function $f: (\mathcal{U}, \mathcal{T}_1) \rightarrow (\mathcal{Z}, \mathcal{T}_2)$ as the identity function. Hence, f is neutrosophic precontinuous but fails to be neutrosophic α -continuous.

Example 10. We have, $\mathcal{U} = \mathcal{Z} = \{a, b, c\}$. The NSs are defined as $Q = \langle (0.6, 0.6, 0.6), (0.3, 0.6, 0.6), (0.3, 0.6, 0.6) \rangle$ and $K = \langle (0.2, 0.3, 0.3), (0.8, 0.6, 0.6), (0.2, 0.3, 0.3) \rangle$. Furthermore, we let $D = \langle (0.2, 0.3, 0.3), (0.2, 0.2, 0.2), (0.3, 0.4, 0.4) \rangle$. The set $\mathcal{T}_1 = \{0_N, 1_N, Q, K\}$ forms a NT on $\mathcal{U}$, while $\mathcal{T}_2 = \{0_N, 1_N, D\}$ is a NT on $\mathcal{Z}$. We define the function $f: (\mathcal{U}, \mathcal{T}_1) \rightarrow (\mathcal{Z}, \mathcal{T}_2)$ with $f(a) = c$, $f(b) = a$, and $f(c) = b$. It is clear that f is neutrosophic β -continuous; however, it is not neutrosophic precontinuous, neutrosophic semicontinuous, nor is it classified as a NPOs or a NSOs.

Theorem 6. A neutrosophic e -compact NTS $\mathcal{U}$ is characterized by the condition that for every family $\{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle: v \in \Lambda\}$ of NeCSs exhibiting the FIP, there exists a nonempty intersection.

Proof 14. Suppose $\mathcal{U}$ is neutrosophic e -compact and $\{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle: v \in \Lambda\}$ is any family of NeCSs in $\mathcal{U}$ such that

$$\bigcap \{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle: v \in \Lambda\} = 0. \quad (7)$$

Therefore, $\bigwedge \{\mu_{Q_v}(r): v \in \Lambda\} = 0$, $\bigwedge \{\sigma_{Q_v}(r): v \in \Lambda\} = 0$ and $\bigwedge \{\gamma_{Q_v}(r): v \in \Lambda\} = 1$. Then,

$$\bigvee \{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle: v \in \Lambda\} = 0. \quad (8)$$

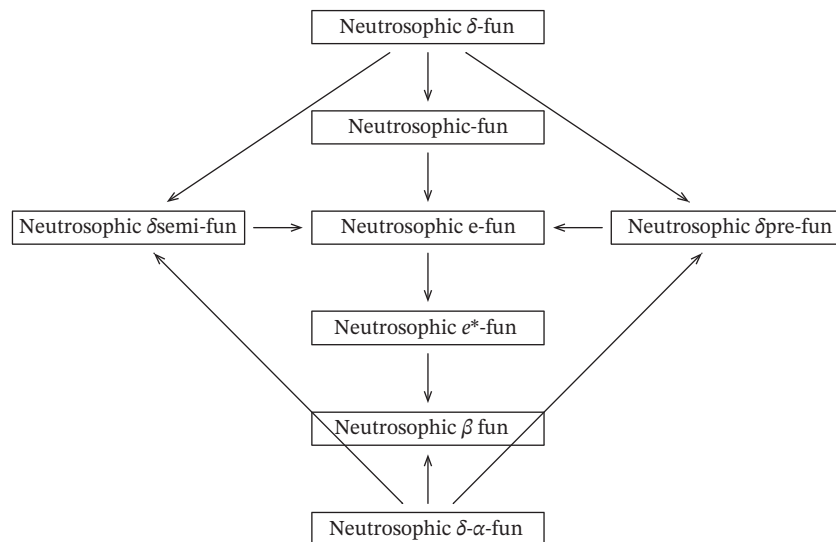


FIGURE 3 | Relation between neutrosophic functions.

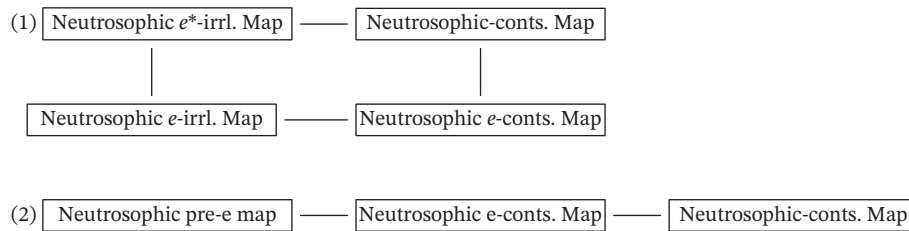


FIGURE 4 | Relation between neutrosophic continuous functions.

As a result, the ensemble $\{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle : v \in \Lambda\}$ constitutes a neutrosophic e -open cover of $\mathcal{U}$. Since $\mathcal{U}$ is classified as neutrosophic e -compact, a finite subcover can be established. If $\mathcal{U}$ is neutrosophic e -compact, there is a finite subcover

$$\{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle : v = 1, 2, 3, \dots, n\}. \quad (9)$$

Then,

$$\bigcup_k^n \{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle : v \in \Lambda = 1\}. \quad (10)$$

Hence, $\bigwedge \{\mu_{Q_v}(r) : v \in \Lambda\} = 0$, $\bigwedge \{\sigma_{Q_v}(r) : v \in \Lambda\} = 0$, and $\bigwedge \{\gamma_{Q_v}(r) : v \in \Lambda\} = 1$. Finally,

$$\bigcap_k^n \{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle : v \in \Lambda = 0\}. \quad (11)$$

We have established that if $\mathcal{U}$ is a neutrosophic e -compact space, then for any collection $\{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle : v \in \Lambda\}$ of NeCSs with an empty intersection, there exists a finite subcollection whose intersection is also empty. Conversely, if $\mathcal{U}$ possesses the FIP, this implies that when the intersection of any family of IFeCSs is empty, then the intersection of every finite subcollection remains empty. Now, consider $\{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle : v \in \Lambda\}$ as a neutrosophic e -open cover of $\mathcal{U}$.

$$\bigcup_k^n \{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle : v \in \Lambda = 1\}. \quad (12)$$

Therefore, $\bigwedge \{\mu_{Q_v}(r) : v \in \Lambda\} = 0$, $\bigwedge \{\sigma_{Q_v}(r) : v \in \Lambda\} = 0$, and $\bigwedge \{\gamma_{Q_v}(r) : v \in \Lambda\} = 1$. Hence,

$$\bigcap_k^n \{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle : v \in \Lambda = 0\}; \quad (13)$$

therefore, the collection $\{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle : v \in \Lambda\}$ represents a set of NeCSs that have no common intersection. Based on this premise, it is possible to identify a finite subset of the family

$$\{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle : v = 1, 2, 3, \dots, n\}, \quad (14)$$

such that

$$\bigcap_k^n \{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle : v \in \Lambda = 0\}. \quad (15)$$

Then,

$$\bigcup_k^n \{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle : v \in \Lambda = 1\}, \quad (16)$$

so $\{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle : v = 1, 2, 3, \dots, n\}$ is a finite subcover of $\{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle : v \in \Lambda\}$. Therefore, $\mathcal{U}$ is neutrosophic e -compact. $\square$

Theorem 7. Define $g: \mathcal{U} \longrightarrow \mathcal{V}$ as a neutrosophic e -continuous mapping between the neutrosophic topologies $\mathcal{U}$ and $\mathcal{V}$. If $\mathcal{U}$ is classified as neutrosophic e -compact, this implies that $\mathcal{V}$ is compact in the fuzzy sense.

Proof 15. This can be compared with the proof found in Theorem 6. $\square$

Remark 5.

1. All neutrosophic e -Lindelöf spaces within NTS (NS in NTS) qualify as fuzzy Lindelöf.
2. All countably neutrosophic e -compact of NTS (NS in NTS) is countably fuzzy compact.
3. All countably neutrosophic e -compact sets in NTS (NS in NTS) are also classified as neutrosophic e -compact.

Theorem 8. When a NTS $\mathcal{U}$ exhibits characteristics of both neutrosophic e -Lindelöf and countably neutrosophic e -compact, it follows that it is neutrosophic e -compact.

Theorem 9. If a neutrosophic space H in a NTS $\mathcal{U}$ exhibits both neutrosophic e -Lindelöf and fuzzy countably neutrosophic e -compact properties, then it can be concluded that H is neutrosophic e -compact.

Proposition 7. Define $g: \mathcal{U} \longrightarrow \mathcal{V}$ as a neutrosophic e -irresolute mapping that transforms the NTS $\mathcal{U}$ onto $\mathcal{V}$. If $\mathcal{U}$ is recognized as neutrosophic e -Lindelöf (or countably neutrosophic e -compact), then it naturally follows that $\mathcal{V}$ is also neutrosophic e -Lindelöf (countably neutrosophic e -compact).

Proof 16. It is comparable to the proof outlined in Theorem 6. $\square$

Theorem 10. Define $\mathcal{U}$ as a neutrosophic e -Lindelöf NTS. In this context, $\mathcal{U}$ is considered countably neutrosophic e -compact if and only if it is also neutrosophic e -compact.

Proof 17. Remark 5 asserts that if $\mathcal{U}$ is defined as neutrosophic e -compact, it also qualifies as countably neutrosophic e -compact. Conversely, let us assume $\{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle : v \in \Lambda\}$ is any neutrosophic e -open cover of $\mathcal{U}$. Given that $\mathcal{U}$ is neutrosophic e -Lindelöf, there exists a countable subcover $\{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle : v = 1, 2, 3, \dots\}$ taken from the original cover. This means that the set $\{\langle x, \mu_{G_i}, \sigma_{G_i} \rangle : i = 1, 2, \dots\}$ can be regarded as a countably neutrosophic e -open cover of $\mathcal{U}$, and, therefore, a subcover $\{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle : v = 1, 2, 3, \dots, n\}$ exists within $\{\langle r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r) \rangle : v = 1, 2, 3, \dots\}$. Consequently, we establish that $\mathcal{U}$ is neutrosophic e -compact. $\square$

Proposition 8. In a NTS, if we consider a NeOS A to be neutrosophic e -Lindelöf, then the set H will be countably neutrosophic e -compact if and only if it is also neutrosophic e -compact.

Proof 18. The demonstration follows a similar approach to that of the earlier theorem. $\square$

Theorem 11. Let us define g as a neutrosophic e_* -open function that operates from the NTS $(\mathcal{U}, \mathcal{T}_1)$ to the NTS $(\mathcal{V}, \mathcal{T}_2)$, with H being an element of $\mathcal{N}(\mathcal{V})$. If H is neutrosophic e -compact in $\mathcal{V}$, then it holds that $g^{-1}(H)$ is also neutrosophic e -compact in $\mathcal{U}$.

Proof 19. Let $K = \{D_\mu: \mu \in \Delta\}$ be a NeOC of $g^{-1}(H)$. Then, $g^{-1}(H) \subseteq \bigcup_{\mu \in \Delta} D_\mu \implies H \subseteq g(\bigcup_{\mu \in \Delta} D_\mu) \implies H \subseteq \bigcup_{\mu \in \Delta} g(D_\mu)$. Since D_μ is a $\mathcal{T}_1$ -NeO set, so $g(D_\mu)$ is a $\mathcal{T}_2$ -NeO set for every $\mu \in \Delta$ as g is a neutrosophic e_* -open function. Therefore, $D = \{(\mu \in \Delta): D_\mu \in K\}$ is a NeOC of H . Since H is neutrosophic e -compact, so D has a finite NeOSC $\{g(D_{\mu_1}), g(D_{\mu_2}), \dots, g(D_{\mu_n})\}$, say. Therefore, $H \subseteq \bigcup_{i=1}^n g(D_{\mu_i}) \implies H \subseteq g(\bigcup_{i=1}^n D_{\mu_i}) \implies g^{-1}(H) \subseteq \bigcup_{i=1}^n D_{\mu_i}$. Therefore, the NeOC K of $g^{-1}(H)$ has a finite NeOSC $\{D_{\mu_1}, D_{\mu_2}, D_{\mu_3}, \dots, D_{\mu_n}\}$. Therefore, $g^{-1}(H)$ is neutrosophic e -compact in $\mathcal{U}$. Then, the proof is done. $\square$

Corollary 2. Assume g is a neutrosophic e_* -continuous function that goes from the NTS $(\mathcal{U}, \mathcal{T}_1)$ to the NTS $(\mathcal{V}, \mathcal{T}_2)$. In the case that H is neutrosophic e -compact in $\mathcal{U}$, it can be concluded that $g(H)$ is also neutrosophic e -compact in $\mathcal{V}$.

Proof 20. Consider $K = \{G_\mu: \mu \in \Delta\}$ as an NeOC of $g(H)$. It follows that the set $g(H)$ lies within the union of the sets G_μ for $\mu \in \Delta$, implying that H is a subset of $g^{-1}(\bigcup_{\mu \in \Delta} G_\mu)$, and thus, H is contained in $\bigcup_{\mu \in \Delta} g^{-1}(G_\mu)$. Since G_μ qualifies as a $\mathcal{T}_2$ -NeO set within $\mathcal{V}$, we can assert that $g^{-1}(G_\mu)$ represents a $\mathcal{T}_1$ -NBO set in $\mathcal{U}$ because g is a neutrosophic e_* -continuous function. As a result, we define the set $D = \{g^{-1}(G_\mu): G_\mu \in K\}$, which serves as a NeOC for H . Considering that H exhibits neutrosophic e -compactness within $\mathcal{U}$, it must have a finite NeOSC $\{g^{-1}(G_{\mu_1}), g^{-1}(G_{\mu_2}), \dots, g^{-1}(G_{\mu_n})\}$. We derive the conclusion that $H \subseteq \bigcup_{i=1}^n g^{-1}(G_{\mu_i})$, which further implies $H \subseteq g^{-1}(\bigcup_{i=1}^n G_{\mu_i})$, leading to $g(H) \subseteq (\bigcup_{i=1}^n G_{\mu_i})$. Thus, the NeOC K associated with $g(H)$ encompasses a finite NeOSC $\{G_{\mu_1}, G_{\mu_2}, G_{\mu_3}, \dots, G_{\mu_n}\}$. This confirms that $g(H)$ is neutrosophic e -compact. $\square$

Corollary 3. Assume g is a neutrosophic e_* -continuous function connecting the NTS $(\mathcal{U}, \mathcal{T}_1)$ with the NTS $(\mathcal{V}, \mathcal{T}_2)$. Should $(\mathcal{U}, \mathcal{T}_1)$ be neutrosophic e -compact, it ensures that $(\mathcal{V}, \mathcal{T}_2)$ retains that same property of being neutrosophic e -compact.

Proof 21. Since g is onto, so $g(\tilde{\mathcal{U}}) = \tilde{\mathcal{V}}$. Let $K = \{G_\mu: \mu \in \Delta\}$ be a NeOC of $\mathcal{V}$. Then, $\bigcup_{\mu \in \Delta} G_\mu = \tilde{\mathcal{V}} \implies \bigcup_{\mu \in \Delta} G_\mu = g(\tilde{\mathcal{U}}) \implies g^{-1}(\bigcup_{\mu \in \Delta} G_\mu) = \tilde{\mathcal{U}} \implies \bigcup_{\mu \in \Delta} g^{-1}(G_\mu) = \tilde{\mathcal{U}}$. Since G_μ is $\mathcal{T}_2$ -NeO set in $\mathcal{V}$, so $g^{-1}(G_\mu)$ is $\mathcal{T}_1$ -NBO set in $\mathcal{U}$ as g is neutrosophic E_* -continuous function. Therefore, $D = \{g^{-1}(G_\mu): G_\mu \in K\}$ is a NeOC of $\mathcal{U}$. Since $\mathcal{U}$ is neutrosophic e -compact, so D has a finite NeOSC $\{g^{-1}(G_{\mu_1}), g^{-1}(G_{\mu_2}), \dots, g^{-1}(G_{\mu_n})\}$, say. Therefore, $\tilde{\mathcal{U}} = \bigcup_{i=1}^n g^{-1}(G_{\mu_i}) \implies \tilde{\mathcal{U}} = g^{-1}(\bigcup_{i=1}^n G_{\mu_i}) \implies g(\tilde{\mathcal{U}}) = \bigcup_{i=1}^n G_{\mu_i} \implies \tilde{\mathcal{V}} = \bigcup_{i=1}^n G_{\mu_i}$. Then, the NeOC K of $\mathcal{V}$ has a finite NeOSC $\{G_{\mu_1}, G_{\mu_2}, G_{\mu_3}, \dots, G_{\mu_n}\}$. Therefore, $\mathcal{V}$ is neutrosophic e -compact. $\square$

5 | Neutrosophic Locally e -Compactness

In this section, we introduce a study to the literature by constructing the concepts of NeO and we study NL e -compact and their prosperities.

Definition 28 (see [32]). An NTS $(\mathcal{U}, \mathcal{T})$ qualifies as NL e -compact if, for each NP $u_{\gamma, \alpha, \beta} \in \mathcal{U}$, there is an ϵ -neighborhood S around $u_{\gamma, \alpha, \beta}$ such that $S(u_{\gamma, \alpha, \beta}) = 1_N$ and S is N e -compact with respect to $\mathcal{U}$.

Remark 6. The preceding definitions outlined in Definitions 22 and 28 clearly indicate that all N e -compact spaces are NL e -compact, although the reverse does not generally hold. Every NL e -compact space is also NL-compact, but the opposite may not be true, as illustrated in the upcoming example.

Example 11. Set Λ to be $[0_N, 1_N]$ and let Q_v be defined as $\{r, \mu_{Q_v}(r), \sigma_{Q_v}(r), \gamma_{Q_v}(r): v \in \Lambda\}$, where we establish that $Q_v(r) = 1/4$ for every $r \in \Lambda$, and $\mu_{Q_v}(r) = 0_N$ for all $r \in \Lambda$. Accordingly, the set $\mathcal{P} = \{0_N, 1_N, Q\}$ forms a NT on Λ . It is also easy to ascertain that $(\Lambda, \mathcal{P})$ is NL-compact though it is not NL e -compact.

Example 12. Assume $(\mathcal{U}, \mathcal{T})$ is an infinite neutrosophic discrete topological space, under these circumstances, $\mathcal{U}$ is NL e -compact though it is not N e -compact.

Theorem 12. Every neutrosophic e -compact space is also classified as a NL e -compact space.

Proof 22. Let $(\mathcal{U}, \mathcal{T})$ represent a neutrosophic e -compact space, and consider $u_{\gamma, \alpha, \beta}$ as a NP in $\mathcal{U}$. Given that $\mathcal{U}$ is neutrosophic e -compact and a NEO set with $u_{\gamma, \alpha, \beta}$ contained in $\mathcal{U}$, it follows that $(\mathcal{U}, \mathcal{T})$ constitutes a NL e -compact space. $\square$

Theorem 13. Assume g signify a neutrosophic e_* -open and e_* -continuous function that transitions from the NTS $(\mathcal{U}, \mathcal{T})$ to the neutrosophic space $(\mathcal{V}, \mathcal{T}_1)$. If the $(\mathcal{V}, \mathcal{T}_1)$ space is NL e -compact, it logically follows that $(\mathcal{U}, \mathcal{T})$ also qualifies as a NL e -compact space.

Proof 23. Let $u_{\gamma, \alpha, \beta}$ represent any NP in $\mathcal{U}$. Additionally, let $z_{\gamma, \alpha, \beta}$ denote the NP in $\mathcal{V}$ such that $g(u_{\gamma, \alpha, \beta}) = z_{\gamma, \alpha, \beta}$. Given that $u_{\gamma, \alpha, \beta} \in \mathcal{U}$ and $\mathcal{V}$ is NL e -compact, then there exists a $\mathcal{T}_1$ -NEO set G where $u_{\gamma, \alpha, \beta} \in D$ and D is neutrosophic e -compact in $\mathcal{V}$. Therefore, if $u_{\gamma, \alpha, \beta} \in D$, it follows that $g(u_{\gamma, \alpha, \beta}) \in D$, which leads to $u_{\gamma, \alpha, \beta} \in g^{-1}(D)$. As g is neutrosophic e_* -open and D is neutrosophic e -compact in $\mathcal{V}$, we can apply Theorem 0.1 to conclude that $g^{-1}(D)$ is neutrosophic e -compact in $\mathcal{U}$. Since g is also a neutrosophic e_* -continuous function, it follows that $g^{-1}(D)$ constitutes a $\mathcal{T}$ -NEO set. Consequently, for every NP $u_{\gamma, \alpha, \beta}$ in $\mathcal{U}$, there exists a $\mathcal{T}$ -NEO set $g^{-1}(D)$ such that $u_{\gamma, \alpha, \beta} \in g^{-1}(D)$ and that $g^{-1}(D)$ is neutrosophic e -compact in $\mathcal{U}$. This establishes that $(\mathcal{U}, \mathcal{T})$ is a NL e -compact space. $\square$

Theorem 14. Consider $(\mathcal{U}, \mathcal{T}_1)$ and $(\mathcal{V}, \mathcal{T}_2)$ as two NTSs. If there exists a NeO bijection function $g: \mathcal{U} \longrightarrow \mathcal{V}$, and g is N e^* -irrl while $(\mathcal{U}, \mathcal{T}_1)$ is NL e -compact, then it follows that $(\mathcal{V}, \mathcal{T}_2)$ must also be NL e -compact

Proof 24. Consider $z \in \mathcal{U}$, where we let $g(z) = r$. Given that $\mathcal{V}$ is NL e -compact, we can identify an ϵ -neighborhood $Q = \{z, \mu_Q(z), \sigma_Q(z), \gamma_Q(z)\}$ centered at $g(z)$. This neighborhood satisfies $Q(g(x)) = \{z, \mu_Q(g(z)), \sigma_Q(g(z)), \gamma_Q(g(z))\} = 1_N$, and Q is N e -compact with respect to $\mathcal{V}$. Now, we ascertain that $g^{-1}(Q)$ is NL-compact in the context of $\mathcal{U}$. Since g is N e^* -irreducible, we

find that $g^{-1}(Q)$ acts as an ϵ -neighborhood for z and that $(g^{-1}(Q))(z) = \langle z, g^{-1}(\mu_Q)(z), g^{-1}(\sigma_Q)(z), g^{-1}(\gamma_Q)(z) \rangle = 1_N$. Thus, for each $z \in \mathcal{U}$, there exists an ϵ -neighborhood $g^{-1}(Q)$ such that $g^{-1}(Q)(z) = 1_N$, confirming that $g^{-1}(Q)$ is N -compact in relation to $\mathcal{U}$. Therefore, $\mathcal{U}$ is classified. $\square$

Theorem 15. Consider g as a function that is both neutrosophic e_* -continuous and neutrosophic e_* -open, established from the NTS $(\mathcal{U}, \mathcal{T})$ to $(\mathcal{V}, \mathcal{T})$. If the space $(\mathcal{V}, \mathcal{T})$ is deemed NL e -compact, then it can be concluded that $(\mathcal{U}, \mathcal{T})$ is also NL e -compact.

Proof 25. Let us define $z_{r,t,s}$ as any NP belonging to $\mathcal{U}$. Additionally, let $u_{\alpha,\beta,\gamma}$ indicate the NP within $\mathcal{V}$ for which $g(z_{r,t,s})$ equals $u_{\alpha,\beta,\gamma}$. Given that $u_{\alpha,\beta,\gamma}$ is part of $\mathcal{V}$ and $\mathcal{V}$ is known to be NL e -compact, it follows that there exists a $\mathcal{T}$ -NeO set G with $u_{\alpha,\beta,\gamma}$ included in D , which is a neutrosophic e -compact set in $\mathcal{V}$. This leads to $u_{\alpha,\beta,\gamma} \in D \implies g(z_{r,t,s}) \in D \implies z_{r,t,s} \in g^{-1}(D)$. As D is both neutrosophic e_* -open and neutrosophic e -compact within $\mathcal{V}$, we can apply Theorem 11 to conclude that $g^{-1}(D)$ is neutrosophic e -compact in $\mathcal{U}$. Additionally, the fact that g is a neutrosophic e_* -continuous function implies that $g^{-1}(D)$ also qualifies as a $\mathcal{T}$ -NeO set. Therefore, for every NP $z_{r,t,s}$ in $\mathcal{U}$, there is a $\mathcal{T}$ -NeO set $g^{-1}(D)$ that contains $z_{r,t,s}$, and $g^{-1}(D)$ is neutrosophic e -compact in $\mathcal{U}$. Hence, $(\mathcal{U}, \mathcal{T})$ is a NL e -compact space. $\square$

Theorem 16. Consider g as a neutrosophic e_* -continuous and neutrosophic e_* -open function mapping from a NT $(\mathcal{U}, \mathcal{T})$ to another NT $(\mathcal{V}, \mathcal{T})$. If the space $(\mathcal{U}, \mathcal{T})$ is neutrosophically locally e -compact, then the space $(\mathcal{V}, \mathcal{T}_1)$ will also be neutrosophically locally e -compact

Proof 26. Consider $u_{\alpha,\beta,\gamma}$ as any NP in the context of $\mathcal{V}$. Due to g being onto, an NP $z_{r,t,s}$ exists in $\mathcal{U}$ such that $(z_{r,t,s}) = u_{\alpha,\beta,\gamma}$. With $z_{r,t,s}$ being in $\mathcal{U}$ and $\mathcal{U}$ being NL e -compact, it follows that there is a $\mathcal{T}$ -NeO set such that $z_{r,t,s} \in D$, where D maintains neutrosophic e -compactness in $\mathcal{U}$. Thus, from the inclusion $z_{r,t,s} \in D$, it can be derived that $g(z_{r,t,s}) \in g(D)$, which implies that $u_{\alpha,\beta,\gamma} \in g(D)$. Given that g is neutrosophic e_* -continuous and D is neutrosophic e -compact in $\mathcal{U}$, we arrive at the conclusion, following 4.18, that $g(D)$ is a neutrosophic e -compact set in $\mathcal{V}$. Moreover, since g operates as a neutrosophic e_* -open function, it confirms that $g(D)$ is a $\mathcal{T}_1$ -NeO set. Consequently, for any NP $u_{\alpha,\beta,\gamma}$ in $\mathcal{V}$, there exists a $\mathcal{T}_1$ -NBO set $g(D)$ such that $u_{\alpha,\beta,\gamma}$ resides in $g(D)$ and $g(D)$ exhibits neutrosophic e -compactness in $\mathcal{V}$. Therefore, we ascertain that $(\mathcal{V}, \mathcal{T}_1)$ is a NL e -compact space. $\square$

6 | Covering Properties via N e -Base

Definition 29. Consider $(\mathcal{U}, \mathcal{T})$ as NTSs, with $NEO(\mathcal{U})$ encompassing all NEO sets found in $\mathcal{U}$. A subset Q of $NEO(\mathcal{U})$ can be regarded as a N e -base for $\mathcal{U}$ if and only if for every $B \in NEO(\mathcal{U})$, it is possible to find a subcollection $\{B_\alpha: \alpha \in \Lambda\}$ within Q such that B is represented as the union $\cup\{B_\alpha: \alpha \in \Lambda\}$, where Λ acts as the index set.

Definition 30. A collection Q° from $NeO(\mathcal{U})$ is deemed a neutrosophic e -subbase (abbreviated as N e -subbase) for $\mathcal{U}$ if the finite intersections of its components yield a N e -base for $\mathcal{U}$.

A subset Q° of $NeO(\mathcal{U})$ is classified as a neutrosophic e -subbase (N e -subbase for short) for $\mathcal{U}$ if the finite intersections of its elements produce a N e -base for $\mathcal{U}$.

Theorem 17. A NT $(\mathcal{U}, \mathcal{T})$ is defined as N e -compact if every collection $K = \{K_v: v \in \Lambda\}$, with $K_v = \{u, \mu_{K_v}(u), \sigma_{K_v}(u), \gamma_{K_v}(u)\}$ for $v \in \Lambda$, constitutes a set of NeCS in $\mathcal{U}$ that has the FIP such that $\bigwedge_{v \in \Lambda} K_v \neq 0_N$.

Proof 27. Consider the collection $\{r, \mu_{K_v}(r), \sigma_{K_v}(r), \gamma_{K_v}(r): v \in \Lambda\}$, which forms a family of NeCS in $\mathcal{U}$ that adheres to FIP. To establish that $\bigwedge_{v \in \Lambda} \{r, \mu_{K_v}(r), \sigma_{K_v}(r), \gamma_{K_v}(r)\} = 0_N$, let us assume instead that $\bigwedge_{v \in \Lambda} \{r, \mu_{K_v}(r), \sigma_{K_v}(r), \gamma_{K_v}(r)\} \neq 0_N$. From this, it follows that $\bigvee \{r, \mu_{K_v}(r), \sigma_{K_v}(r), \gamma_{K_v}(r)\} = 1_N$. This suggests that $\{r, \mu_{K_v}(r), \sigma_{K_v}(r), \gamma_{K_v}(r): v \in \Lambda\}$ serves as a NeOS cover for $\mathcal{U}$. With $\mathcal{U}$ being N e -compact, there must exist a finite subset $\{\ll r, \mu_{K_v}(r), \sigma_{K_v}(r), \gamma_{K_v}(r) \gg : v = 1, 2, \dots, n\}$ such that $\bigvee_{v=1}^n \bar{K}_v = 1_N$, which implies $\bigwedge_{v=1}^n K_v = 0_N$. This result contradicts the FIP of the family, confirming that $\bigwedge_{v=1}^n K_v \neq 0_N$.

Alternatively, consider $K = \{K_v: v \in \Lambda\}$ as a NeOSs cover of $\mathcal{U}$ while assuming that $\mathcal{U}$ is not N e -compact. In this scenario, it follows that no finite subcollection of K can serve as a cover for $\mathcal{U}$. If we have $\bigvee_{v=1}^n K_v \neq 1_N$, this suggests that $\bigwedge_{v=1}^n K_v \neq 0_N$. Since the family $K = \{K_v: v \in \Lambda\}$ adheres to the FIP, we can deduce that $\bigwedge_{v=1}^n \bar{K}_v \neq 0_N$, which means that $\bigwedge_{v=1}^n K_v \neq 1_N$. This results in a contradiction, affirming that $\mathcal{U}$ is indeed N e -compact. $\square$

Remark 7. As demonstrated in Theorem 18, it is apparent that when an NTS $(\mathcal{U}, \mathcal{T})$ is neutrosophic $e - C_2$, the notions of neutrosophic e -compactness and neutrosophic countably e -compactness align perfectly.

Theorem 18. Assume that $(\mathcal{U}, \mathcal{T})$ is a neutrosophic countably e -compact space. If $\mathcal{U}$ is identified as neutrosophic $e - C_2$, then it can be asserted that $\mathcal{U}$ is neutrosophic e -compact.

Proof 28. Consider the set $G = \{F_j: j \in \Lambda\}$ as a NeO-cover of $\mathcal{U}$. Since $\mathcal{U}$ qualifies as a neutrosophic $e - C_2$, there is a countable Ne-base $Q = \{Q_k: k = 1, 2, 3 \dots\}$ associated with $\mathcal{U}$. Each member $F_j \in G$ can consequently be represented as a countable union of elements from Q , signifying that for every $F_j \in G$, we have $F_j = \bigcup_{n=1}^{j_0} Q_{k_n}$, with $Q_{k_n} \in Q$ and j_0 possibly extending to infinity. Thus, $Q_0 = \{Q_{k_n}\}$ constitutes an NeO-cover of $\mathcal{U}$. Additionally, since Q_0 is a subset of B , it maintains its countability, establishing it as a countable NeO-cover of $\mathcal{U}$. Therefore, we conclude that $\mathcal{U}$ is countably e -compact, implying that Q_0 possesses a finite NeOSC, denoted as Q' . By design, every element of Q' lies within some F_j belonging to G , thereby confirming that these collections F_j indeed compose a finite NeO-cover for $\mathcal{U}$. Consequently, the NeO-cover G encompasses a finite NeOSC, establishing that $\mathcal{U}$ is neutrosophic e -compact. Thus, the proof is complete. $\square$

Lemma 2. Let Q be an N e -base for NTS $(\mathcal{U}, \mathcal{T})$. Therefore $\mathcal{U}$ is neutrosophic e -compact if and only if for each NeO-cover of $\mathcal{U}$ by the N e -base of Q has a finite NeOS-comp.

Proof 29. We will prove the sufficient part since the necessary part is obvious.

Suppose that $Q = \{Q_\lambda: \lambda \in \Lambda\}$ is the N e -base, and let $K = \{R_\alpha: \alpha \in \Lambda\}$ be an NEO-comp of $\mathcal{U}$. Therefore, each member R_α of K is the arbitrary union of Q and the totality of such

members of Q is evidently an NEO-comp of $\mathcal{U}$. By using the hypothesis, this arbitrary union of Q has a finite NEOSC put $G = \{Q_{\lambda_i} : i = 1, 2, 3 \dots, n\}$. Clearly, for every Q_{λ_i} in G , there is a G_{α_i} in K such that $Q_{\lambda_i} \subseteq G_{\alpha_i}$. This implies the finite subcollection $\{G_{\alpha_i} : i = 1, 2, 3 \dots, n\}$ of K is an NeO-cover of $\mathcal{U}$; this mean the NeO-cover K of $\mathcal{U}$ has a finite NeOSC. Then, $\mathcal{U}$ is neutrosophic e -compact. $\square$

Theorem 19. Suppose that $(\mathcal{U}, \mathcal{T})$ is a neutrosophic $e - C_2$, then it is neutrosophic e -Lindelöf.

Proof 30. Let $G = \{F_j : j \in \Lambda\}$ be any NEOC of $\mathcal{U}$. Since $\mathcal{U}$ is neutrosophic $e - C_2$, so there exists a countable N e -base $Q = \{Q_k : k = 1, 2, 3 \dots\}$ for $\mathcal{U}$. Therefore, every $F_j \in G$ can be defined as the countable union of elements of Q ; briefly for every $F_j \in G$, we have $F_j = \bigcup_{n=1}^{j_0} Q_{k_n}$, where $Q_{k_n} \in Q$ and j_0 may be infinity. Suppose that $Q_0 = \{Q_{k_n}\}$. Since Q_0 is an NeO-cover of $\mathcal{U}$. Moreover, Q_0 is countable as $Q_0 \subseteq Q$. Then, Q_0 is a countable NeO-cover of $\mathcal{U}$. By construction, every element of Q_0 is contained in one member F_j of G so that the collections F_j of G form a countable NeOS-cover of $\mathcal{U}$. This implies the NeO-cover G of $\mathcal{U}$ has a countable NeOSC. Then, $\mathcal{U}$ is neutrosophic e -Lindelöf. $\square$

Theorem 20. Consider ψ to be an N e -subbase of a NTS $(\mathcal{U}, \mathcal{T})$. The space $\mathcal{U}$ qualifies as neutrosophic e -compact if, and only if, every collection of NeC sets originating from ψ^c with the FIP results in a nonempty intersection.

Proof 31. Essential part: Immediate from Theorem 17.

Sufficient: Assume that $\mathcal{U}$ is not e -compact. According to Theorem 17, there exists a collection $K = \{Q_\gamma : \gamma \in \Lambda\}$ of NeC sets within $\mathcal{U}$ that exhibit the finite intersection property, implying that $\bigcap_{\gamma \in \Lambda} R_\gamma = \phi$. The set $A = \{K\}$, which contains all such collections K , can be organized using the traditional inclusion relation ($\subseteq$). Consequently, collection T will have an upper limit. Thus, by the lemma concerning facts, there exists a maximal collection of all these collections K . Define $Y = \{C_\gamma : \gamma \in \Lambda\}$ as the maximal collection. This collection Y evidently possesses the following characteristics:

1. $\phi \notin \mathcal{U}$.
2. $Q \in \mathcal{U}, Q \subseteq K \implies K \in \mathcal{U}$.
3. $Q, K \in \mathcal{U} \implies Q \cap K \in \mathcal{U}$.
4. $\cap(\mathcal{U} \cap \psi^c) = \phi$.

The existence of Property (4) clearly contradicts the hypothesis, thereby indicating that $\mathcal{U}$ is neutrosophic e -compact. $\square$

7 | Conclusion

Create more complex fuzzy continuous mappings, such as fuzzy weakly continuous functions and fuzzy e -continuous functions. Examine the interactions between fuzzy e -compact sets, fuzzy closed sets, and fuzzy separation axioms. In order to better model uncertainty, develop new characterizations of fuzzy spaces with relaxed compactness behavior.

Many researchers have contributed to the examination of e -open sets, e -continuity, e -compactness, neutrosophic countably e -compactness, neutrosophic e -Lindelöfness, and other related aspects in the field of topological spaces. Our findings confirm

that all neutrosophic e -compact spaces qualify as neutrosophic compact spaces; however, this is not mutually applicable in the opposite direction. Additionally, we have shown that if g is a neutrosophic continuous or e -continuous function mapping a neutrosophic e -compact (or countably e -compact, e -Lindelöf) space $(\mathcal{U}, \mathcal{T}_1)$ to a NTS $(\mathcal{V}, \mathcal{T}_2)$, this implies that $(\mathcal{V}, \mathcal{T}_2)$ will be a neutrosophic compact (or countably compact, Lindelöf) space. We have further defined and investigated a range of properties related to neutrosophic local e -compactness. In the concluding section, we presented the notions of N e -base, e -subbase, and neutrosophic $e - C_2$, while also analyzing their pertinent properties. We concluded that if a NTS is categorized as neutrosophic $e - C_2$, then an equivalence exists between neutrosophic e -compactness and neutrosophic countably e -compactness.

We contribute to the literature by constructing the concepts of NO (resp., $N \delta$ PO, $N \delta$ O, $N \delta$ SO, NeO, NS e^* O, and $N \beta$ O) and we study open map, continuity, and irresoluteness. Then, we study the relation between all the concepts and there prosperities. Also, we introduce a study to the literature by constructing the concepts of NeO and we study NL e -compact, N e -base, and their prosperities.

7.1 | Future Research Directions Section

The notion of neutrosophic e -sets, which expands upon classical and fuzzy set theories by integrating the ideas of external behavior and indeterminacy, provides a versatile basis for further developments into more sophisticated uncertainty frameworks.

Developing computational methods for determining coverings in large or complex NTSs. Applying the theoretical results to practical problems in decision-making, knowledge-based systems, and data analysis under uncertainty. Exploring additional topological properties, such as compactness and separation axioms, within the neutrosophic context to broaden the scope of applicability.

A notably encouraging path is the integration of neutrosophic e -sets into neutrosophic rough sets, which permits the reconfiguration of approximation spaces using δ -open or δ -interior concepts to more effectively tackle vagueness and boundary issues in the presence of external uncertainty.

The notion of neutrosophic e -sets can be notably broadened to include more complex topological structures, such as hypertopology, multitopology, and superhypertopology, thereby unlocking fresh possibilities for the generalized modeling of uncertainty. In hypertopological spaces, which prioritize sets over individual points, and where the topology is based on the hyperspace of subsets, neutrosophic e -sets present an advanced method for addressing set-valued uncertainty by incorporating external openness and indeterminacy within hyperspace configurations. Likewise, multitopological spaces also gain advantages from this theoretical framework.

Topological structures based on quadripartitioned NSs (e.g., Das et al. [34]) are possible expansions of our framework that include additional integral measures and open-set types. Recent developments in HyperNeutrosophic and SuperHyperNeutrosophic Sets, which build upon multineutrosophic frameworks, allow for further generalization [35]. Furthermore, intriguing approaches for modeling heterogeneous uncertainty are provided by Plithogenic and HyperPlithogenic Sets and

Topologies [36]. We cite these seminal publications to support vocabulary and theoretical underpinnings when discussing innovative entities like “superhypertopology”.

Nomenclature

$\mathbb{Q}$	Set of real numbers
$\mathbb{N}$	Set of natural numbers
$\emptyset$	Empty set
$\subseteq$	Subset relation
U, V, Z	Generic topological spaces
A°	Interior of set A
$\bar{A}$	Closure of set A
IF	Intuitionistic fuzzy
NS	Neutrosophic set
NRS	Neutrosophic rough set
N δ Os	Neutrosophic δ -open set
N δ POs	Neutrosophic δ -pre open set
N δ SOs	Neutrosophic δ -semi open set
N $\delta\alpha$ Os (or) N α Os	Neutrosophic δ - α -open (or) α -open set
N e^* Os	Neutrosophic e^* -open set
N β Os	Neutrosophic β -open set
NL e -compact	Neutrosophic locally e -compact
N e -base	Neutrosophic e -base
N δ -map	Neutrosophic δ -map
N δ P-map	Neutrosophic δ -pre map
N δ S-map	Neutrosophic δ -semi map
N $\delta\alpha$ -map (or) N α -map	Neutrosophic δ - α -map (or) α -map
N e^* -map	Neutrosophic e^* -map
N e -map	Neutrosophic e -map

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Conflicts of Interest

The author declares no conflicts of interest.

Data Availability Statement

The data used to support the findings of this study are available from the corresponding author upon reasonable request.

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